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RESEARCH REPORT 185

MHD CHARACTERISTICS AND SHOCK WAVES

by

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ABSTRACT

A review of the theory of MHD characteristics and shock waves is presented. Primary emphasis is placed on a physical discussion of the three characteristic modes and the jump conditions for the two types of shock waves which can exist. Brief discussions of shock structure, applications of the theory, and the range of applicability of the continuum equations are also given.

Prepared as a chapter in a book on plasma physics to be edited by Wulf B. Kunkel.

SECTION I

INTRODUCTION

In ordinary gas dynamics the theory of nonlinear flows which are either time-dependent or involve supersonic velocities is well developed. Such flows tend to contain planes across which the significant changes in gas conditions occur almost discontinuously. These discontinuities which are called shock waves, become so thin (of the order of a few mean free paths) that the dissipation rates within them become extremely large. One of the prime reasons for interest in shock waves is that they provide a mechanism for converting flow kinetic energy ahead of the wave into thermal energy behind the wave and thus provide a controlled means of producing high temperature gases. Our ability to deal with flows containing shock waves is greatly facilitated by two facts. First, the changes in flow properties across a shock are independent of the detailed structure of the shock wave. Thus, the conservation laws yield algebraic relations which define conditions behind a shock in terms of its velocity and the conditions ahead. Secondly, the shock waves are usually (when the typical flow dimension is large compared to a mean free path) so thin compared to the overall flow field that they can be treated as sheets across which flow properties change discontinously.

In the remainder of the flow field, although the fluid properties may change by large amounts, the gradients will be small. These regions may be treated by a method which is essentially a generalization to the nonlinear case of concepts developed in the description of linear wave propagation in nondispersive media. In the propagation of electromagnetic signals in vacuum or in acoustics a disturbance of arbitrary shape at one instant of time will have the same shape at a later time but will be displaced by a distance equal to the product of the propagation speed and the time difference (Fig. 1(a)). Now, if this disturbance propagated through a medium in which the propagation speed varied with position, due to gradual changes in the index of refraction or gas temperature it would become distorted as it propagates. For a nondispersive medium (i. e., a medium in which the propagation speed is independent of the wave length of the disturbance) the changes in shape could be obtained by considering the disturbance to be composed of a large number of small step functions and following each step on a distance-time diagram (Fig. 1(b)). The slope of the trajectory of each step is equal to the local propagation speed. This technique would not work for a dispersive medium since the different Fourier components of the step functions would be spread out in space. It is significant to note, however, that the applicability of the above method is not restricted to small variations in the propagation speed. Thus, arbitrarily large variations in propagation speed can be taken into account.

For the nonlinear gas dynamic problem, let us first remember that a monatomic gas is a nondispersive medium for wave lengths long compared to the mean free path.* As an example, we imagine a long pipe containing an initially uniform gas, in which a disturbance is generated by moving a piston at one end of the pipe. This disturbance may be viewed as consisting of a large number of small amplitude step function waves all propagating away from the piston. Each wave will propagate through the fluid at a velocity equal to the sound speed determined by the local fluid properties. Since the medium is nonlinear the propagation speed of a particular wave will depend upon the changes produced by the previous waves. Thus different waves will propagate at different velocities and the disturbance will change its shape with time. As lone as the waves do not cross, the number of waves preceding any particular wave is independent of time. This implies that conditions ahead of the wave and, therefore, its propagation speed are constant. The distortion of the disturbance with time is, therefore, determined by following each point on the disturbance along a straight line in the distance-time plane. (Fig. 1(c)). Since the velocity for each point is different some of these lines will diverge while others converge. Note that this is fundamentally different from the linear case of wave propagation through a spatially varying medium. In that case the trajectories of all the waves composing the disturbance are parallel at a particular point in space. In the nonlinear case the converging lines can intersect. They cannot, however, cross one another since the wave from behind is only overtaking the previous wave because of the changes in fluid properties produced by the first wave. Thus, if the second wave did go ahead of the first wave it would then propagate more slowly. Thus, the converging waves tend to pile up and produce a large amplitude discontinuity or shock wave.

The concepts which have been described lead in the case of converging waves to a description of the formation of shock waves. In the case of diverging waves they can be used to describe flow changes of arbitrarily large amplitude. Thus far, we have discussed only the situation in which all waves are propagating in one direction. In the more general case, for example, after the disturbance reflects from the far end of the pipe, there will be waves propagating in both directions. In this case the general concept of following the trajectories of small step function waves is still useful. However, it usually requires numerical integration along a two-dimensional grid.

Mathematically, this description of gas dynamics flows is a special case of the theory of characteristics which is applicable to certain types of hyperbolic partial differential equations. It was first noticed by Friedrichs[1] that the mathematical theory was also applicable to the magnetohydrodynamic

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* For polyatomic gases we would have to restrict ourselves to time scales that are either very long or very short compared to the relaxation times for internal degrees of freedom.

equations. Much of the basic work on the description of nonlinear wave propagation in magnetohydrodynamics was covered in his initial paper. The purpose of the present chapter is to review the present status of this theory. In the presentation given here we will attempt to emphasize a physical description of the subject. In this way we hope that it may be somewhat easier for the reader who is not familiar with the formal mathematical theory of characteristics to become acquainted with its application to gasdynamics and magnetohydrodynamics. Although we will not assume a knowledge of the theory of characteristics in ordinary gasdynamics, the reader may wish to refer to some of the following discussions on the subject.[2]

Since the theory is an extension of the linear analysis we will begin by deriving the properties of the linear waves in Section II. We may note at this point that the magnetohydrodynamic case will be considerably more complex than the ordinary gas dynamic case. The presence of the magnetic field in the plasma defines a direction within a plasma. Thus, the wave propagation speed as well as its properties will depend upon the direction in which the wave is propagating relative to the magnetic field. Furthermore, the presence of the magnetic field requires the existence of three distinct propagation modes of small amplitude waves, as opposed to only one sound speed which exists in the absence of the field. A simple explanation of this can be given if we imagine that the waves are produced by a piston which forms one boundary of the plasma. The piston has three degrees of freedom and therefore we would expect that a wave mode is required for each degree of freedom. However, in the absence of a magnetic field motions of the piston parallel to its plane are not observed in the gas (except within a diffusion layer immediately adjacent to the piston). Thus, for an ordinary gas, only motions of the piston normal to itself produce propagating waves which are observable at appreciable distances from the piston. However, in the case of a plasma if the magnetic field has a component normal to the plane of the piston and the piston is a conductor, the motions of the piston in its plane require a corresponding motion of the field lines. Since the plasma is frozen to the field lines it must move with the piston. As a result such motions also produce waves and therefore a wave exists for each degree of freedom of the piston motion.

In Section III, we will discuss the extension from the linear to the nonlinear case. In particular, we will show that two of the modes of linear propagation lead to the formation of shock waves. The third mode on the other hand, is linear even for large amplitude. As a result this third mode does not steepen to form shock waves but even for large amplitude waves the shape is maintained as the wave moves through the plasma.

In Section IV we will discuss the resulting two types of shock waves. We may note at this point that whereas ordinary gasdynamics shock waves represent an interchange of energy between flow kinetic energy and thermal energy, in the plasma magnetic energy is also present. Thus, there are situations in which some of the flow kinetic energy is converted into magnetic energy reducing the amount of thermal energy which is produced. On the other hand, there are also shock waves for which the basic energy source is the magnetic field and thus the magnetic energy is converted into thermal energy and flow kinetic energy.

In Section V, we will discuss briefly the internal structure of shock waves. Finally, in Section VI we will discuss the application of the theory in two cases; first to the production of high temperature plasma samples in the laboratory, and second to the rate at which magnetic energy can be converted to plasma energy. The latter case has possible interest in several astrophysical situations.

Throughout the body of the text (with the exception of a portion in Section V) we will assume that the fluid can be described by the magnetohydrodynamic equations. The delineation of the range of plasma conditions over which this is a valid assumption is at present not clearly understood. Over an interesting but somewhat limited range of temperatures and densities the validity can be justified with reasonable certainty on the basis of rapid maxwellization by binary collisions. There are, however, theoretical arguments as well as some experimental evidence which strongly suggest that the actual range of validity is considerably larger, including very high temperature plasma conditions in which binary collisions are rare provided only that the gyro radii of the ions and the Debye length of the plasma are small compared to the length in which the overall properties of the flow field change significantly. Further remarks on the justification of the validity of the magnetohydrodynamic equations are given in the Appendix.

In the presentation which follows we will in general not give detailed references to the specific papers in which a particular conclusion was originally presented. An extensive but far from complete bibliography is, however, included. This chapter does not contain any concepts which have not been published elsewhere, however, the presentation may have been modified.

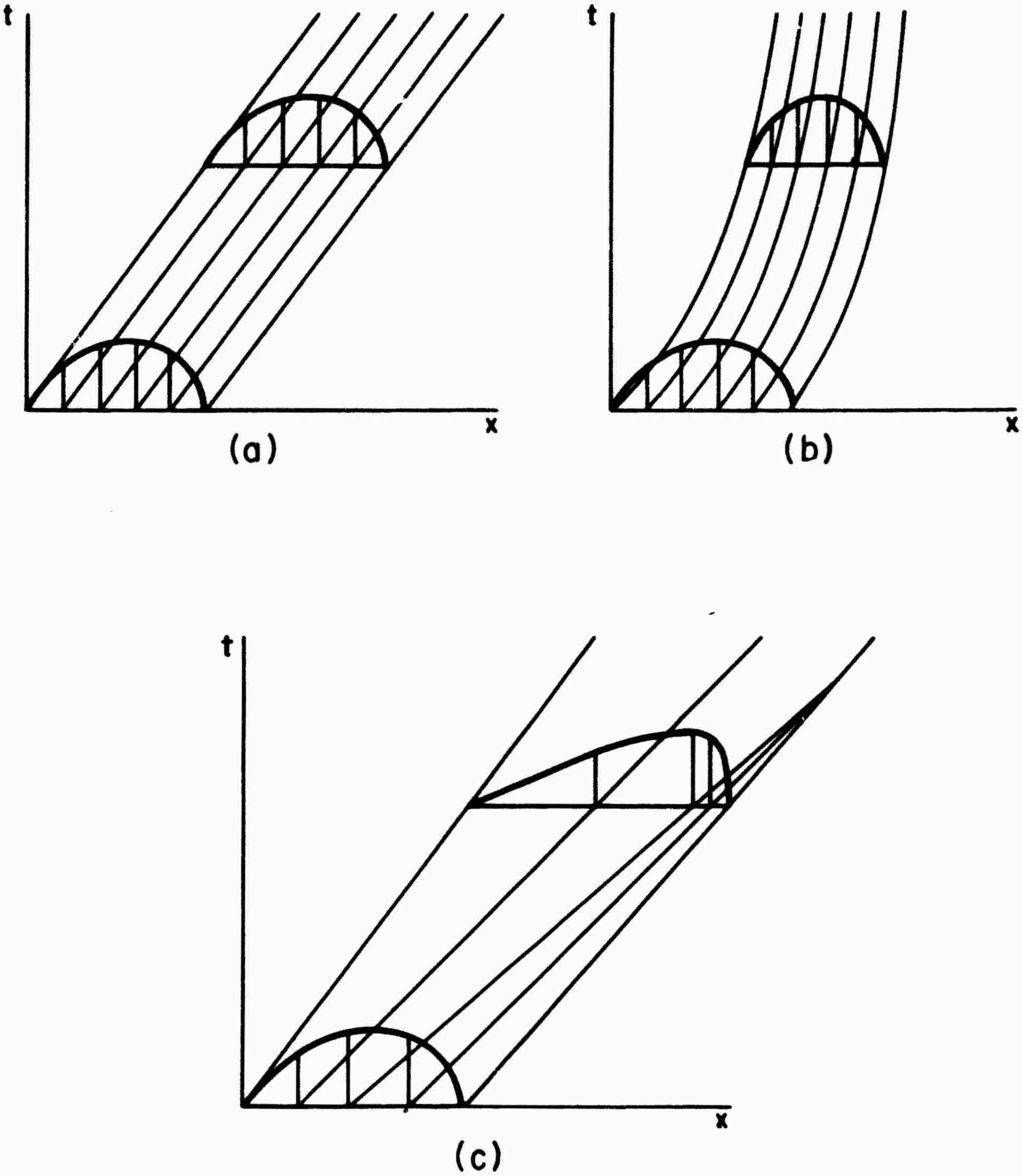

Fig. 1 Illustration of propagation of an arbitrary pulse in several cases. a) For linear waves in a uniform medium, the pulse shape is retained. b) For linear waves in a medium of variable propagation speed, the pulse becomes somewhat distorted, however, discontinuities do not arise. c) For a nonlinear pulse in an initially uniform medium, changes in propagation speed can lead to the formation of discontinuities or shock waves.

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SECTION II

LINEAR WAVES

The magnetohydrodynamic equations for a nondissipative medium (i.e., infinite electrical conductivity, and vanishing viscosity and heat conductivity) are:

Continuity $$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \qquad \text{(II-1)}$$

Momentum $$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p - \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} = 0 \qquad \text{(II-2)}$$

Induction $$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0 \qquad \text{(II-3)}$$

Entropy $$\frac{\partial (p/\rho^{\gamma})}{\partial t} + \mathbf{v} \cdot \nabla (p/\rho^{\gamma}) = 0 \qquad \text{(II-4)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(II-5)}$$

where ρ is the density, $\mathbf{v}$ the fluid velocity, p the plasma pressure, $\mathbf{B}$ is the magnetic field intensity and γ is the ratio of specific heats, C_p/C_v. The neglect of dissipative terms requires that the gradients be small. Thus, these equations will not apply to shock waves and for cases in which steepening occurs the resulting shock waves must be isolated and treated separately.

As indicated earlier, the theory of characteristics describes the nonlinear flow as composed of small amplitude step function waves. We shall therefore begin by examining the small amplitude waves which result from Eqs. (II-1) through (II-5). We also noted that the scheme will only apply to a nondispersive medium. It can be seen by examining the equations that it is impossible to form either a basic length or a basic time from the quantities in the equations. Therefore, if we examined the linear waves by the more usual method of assuming sinusoidally shaped waves, we would find that the phase velocity of these waves is independent of wavelength, i.e., the medium is nondispersive. The extension to the nonlinear case is more direct if we begin by deriving the linear wave properties in terms of step functions.

It is convenient to analyze the wave in a coordinate system in which the wave is stationary. In this coordinate system the fluid will of course be moving on both sides of the wave. The fluid velocity ahead of the wave must be equal and opposite to the propagation speed of the wave relative to the fluid in order for the wave to be stationary. We will denote this velocity by c to indicate that it is the characteristic propagation speed of the wave. Let us further orient our coordinate system such that the wave normal is in the x-direction, and such that the magnetic field ahead of the wave is in the x-y plane, i.e., $B_z = 0$ ahead of the wave. In our coordinate system the time derivatives are zero and the ∇ operator reduces to a derivative in the x-direction. If we now integrate Eqs. (II-1) through (II-5) across the wave, we obtain relations for the changes in the flow properties across the wave. Keeping terms only to first order in the changes, we thus obtain for the continuity equation

$$\delta (\rho v_x) = - c\delta\rho + \rho\delta v_x = 0 \qquad \text{(II-6)}$$

the three components of the momentum equation become

$$-\rho c \delta v_x + \delta p + \frac{B_y \delta B_y}{4\pi} = 0 \qquad \text{(II-7)}$$

$$\rho c \delta v_y + \frac{B_x \delta B_y}{4\pi} = 0 \qquad \text{(II-8)}$$

$$\rho c \delta v_z + \frac{B_x \delta B_z}{4\pi} = 0 \qquad \text{(II-9)}$$

The components of Eq. (II-3) become

$$B_y \delta v_x - c \delta B_y - B_x \delta v_y = 0 \qquad \text{(II-10)}$$

$$c \delta B_z + B_x \delta v_z = 0 \qquad \text{(II-11)}$$

where we have already made use of Eq. (II-13) below. Equation (II-4) becomes

$$c(\delta p - \frac{\gamma P}{\rho} \delta \rho) = 0 \qquad \text{(II-12)}$$

and finally Eq. (II-5) becomes

$$\delta B_x = 0 \qquad \text{(II-13)}$$

The unknowns in the above equations are the speed of propagation of the wave, c, and the various quantities which change across the wave. Counting equations and unknowns in this manner we find one more unknown than equations. The equations are, however, homogeneous in the unknowns corresponding to the changes across the wave. Therefore, as one would expect for the linearized case, the differential equations do not determine the amplitude of the wave. We do, however, have the appropriate number of equations and unknowns to determine the speed of propagation of the wave and the changes of flow properties across the wave in terms of the wave amplitude. We shall proceed to do this in a somewhat disorderly fashion by observing that some of the equations form sub-sets whose solutions can be easily determined.

Entropy Discontinuities

We will first discuss a solution which is not really a wave, since it corresponds to zero propagation speed. We may observe by inspection that substituting c = 0 into Eqs. (II-6) through (II-13) is a solution which allows a change in density but requires that there be no change in all of the other flow properties, i.e., velocity, magnetic field, and gas pressure. This is a hydrostatic equilibrium corresponding to having different density, entropy, and temperature on the two sides of the discontinuity, but maintaining the pressure constant. Since there is no flow through the wave, this result is of course, consistent with our entropy conservation law Eq. (II-4), which stated that following a fluid element, the entropy was conserved. Thus, if initially adjacent elements of fluid have different entropies, this discontinuity in entropy will be maintained. A practical case in which entropy discontinuities are of importance is a flow situation in which shock waves exist whose strength is not constant. The entropy changed across a shock depends upon its strength. Thus, fluid elements going through the shock at slightly different times will have slightly different entropies.

Since we have determined the only solution which can exist for zero propagation speed, we may, in looking for the other solutions, assume that the propagation speed is nonzero. Thus, we may take the bracket in Eq. (II-12) to be equal to zero. This corresponds simply to the statement that if there is a flow through the wave and if entropy is conserved along a streamline, then the entropy on both sides of the wave must be the same.

Intermediate Waves

As noted earlier, we expect three propagating waves corresponding to the three degrees of freedom of motions of the boundary. In this sub-section we will derive the properties and propagation speed of one of these modes. It so happens that the three modes can be conveniently classified as fast, intermediate, and slow according to the magnitudes of their speeds of propagation. The wave to be discussed in this sub-section corresponds, as will be shown later, to the intermediate propagation speed.

The only equations which contain δv_z and δB_z are Eqs. (II-9) and (II-11). Since these equations also do not contain the changes in any of the other properties, we may solve them separately. Doing this, we obtain for the propagation speed

$$c_i^2 = \frac{B_x^2}{4\pi\rho} \tag{II-14}$$

where the subscript i indicates the intermediate wave. Substituting this result back into the full set of equations, we obtain the following relations for the changes in flow properties across the wave.

$$\delta v_z = \pm \frac{\delta B_z}{\sqrt{4\pi\rho}} \tag{II-15}$$

$$\delta v_x = \delta\rho = \delta p = \delta B_y = \delta v_y = 0$$

The sign in the top equation depends upon whether the direction of propagation is parallel or antiparallel to the normal component of magnetic field. Since δv_x is zero this wave is purely transverse. Also, since the change in magnetic field is perpendicular to the original field, there is to first order only a change in the direction of the magnetic field, but no change in magnitude. (We shall see later that a large amplitude intermediate wave also changes only the direction and not the magnitude of the magnetic field.) It is interesting to note that there are no changes in the thermodynamic variables across such a wave. The only changes are in the tangential velocity and the direction of the magnetic field.

A simple physical explanation of this wave is frequently given in terms of a vibrating string. Since the wave is purely transverse, one would expect that the wave propagation speed should be the square root of the tension divided by the density. Since the tension in the direction of the wave normal is $B_x^2/4\pi$, we see that this description is in complete agreement with the result given in Eq. (II-14). It should, however, be emphasized that, whereas the vibrating string can vibrate in any direction perpendicular to the string, the intermediate wave corresponds to changes in magnetic field only in the z-direction. Thus, the ordinary vibrating string really corresponds to two modes which have the same propagation velocity. In the plasma, only one of these modes corresponding to magnetic field changes perpendicular to the original magnetic field gives rise to the intermediate propagation speed. Waves in which a change in B_y also exists will give rise to some longitudinal stresses, and as a result their propagation speed is modified. Such waves are then either the fast or the slow mode.

Fast and Slow Waves

The propagation speed of the remaining two modes can be obtained by eliminating the quantities which change across the wave from Eqs. (II-6, II-7, II-8, II-10, and II-12). The resulting relation for the propagation speed may be written in the following two completely equivalent forms.

$$(c^2 - a^2)(c^2 - b_x^{\,2}) = c^2 b_y^{\,2} \qquad \text{(II-16)}$$

$$(c^2 - a^2)(c^2 - b^2) = a^2 b_y^{\,2} \qquad \text{(II-17)}$$

where we have introduced shorthand notations for the ordinary sound speed and for a vector whose magnitude is equal to the Alfven speed and whose direction is parallel to the magnetic field.

$$a = \sqrt{\frac{\gamma p}{\rho}} \qquad \underset{\sim}{b} = \frac{\underset{\sim}{B}}{\sqrt{4\pi\rho}} \qquad \text{(II-18)}$$

The above equation is seen to be bi-quadratic in the propagation speeds, and thus corresponds to two modes each of which can propagate in two directions. Let us first observe some of the properties of the propagation speeds resulting from this relation. We will return to physical interpretation of these results somewhat later when we discuss the changes which occur across the waves.

Since, in both forms of the dispersion relation which have been written, the right-hand sides are positive definite, it follows that the quantities in parentheses on the left-hand side must either be both positive or both negative. It, therefore, follows from Eq. (II-16) that for one of the solutions, the square of the propagation speed must be greater than both a^2 and $b_x^{\,2}$, and for the other solution it must be less than both of these quantities. Since, by definition, the slow speed (subscript s) is the slower of these, it follows that

$$c_s^2 \leq b_x^2, \quad c_s^2 \leq a^2 \tag{II-19}$$

A somewhat more stringent condition on the fast propagation speed (subscript f) is obtained from Eq. (II-17).

$$c_f^2 \geq b^2, \quad c_f^2 \geq a^2 \tag{II-20}$$

Since, in the present notation the intermediate speed is b_x, we note from the above relations that the wave which we have labelled fast always travels at a speed greater than or equal to the intermediate speed, while the one which has been labelled slow is always slower than or equal to the intermediate speed, thus justifying naming the wave modes in terms of their relative propagation speeds.

The solutions of this dispersion relation have been plotted in Fig. 2 for several ratios of a to b. These plots are given in the form suggested by Friedrichs. The magnetic field direction is taken to be horizontal. For any point on the lines, the distance from the origin is proportional to the velocity of the wave, and the angle which the line connecting the origin to the point makes with the axis corresponding to the magnetic field is the direction of the wave normal relative to the magnetic field. For the fast mode, the propagation speed is relatively insensitive to the direction of propagation. For propagation perpendicular to the magnetic field, the propagation speed is $\sqrt{a^2 + a^2}$. For propagation along the magnetic field, the propagation speed is either a or b depending upon which is larger. The intermediate propagation speed is also shown and corresponds to two circles of radius b/2 whose line of centers lies in the magnetic field direction and are tangent at the origin. Since the slow propagation speed is less than the intermediate speed, the slow speed must be found inside of these circles. Thus, we see that both the slow and the intermediate speeds are zero for propagation perpendicular to the magnetic field. For propagation along the magnetic field, the slow speed is either a or b depending upon which is smaller. We note from the above remarks that for propagation along the magnetic field, the intermediate speed is always equal to either the slow or the fast speed. Thus, for $a < b$, the intermediate and fast speeds are equal for this direction of propagation, while for $a > b$, the intermediate and slow speeds are equal.

Let us examine briefly the limiting values of the solutions of the dispersion relation for the cases of $a \gg b$ and $a \ll b$. These correspond to gas pressure much larger than and much less than the magnetic pressure respectively. For $a \gg b$, the fast propagation speed must also be much greater than b, and therefore, from Eq. (II-16) we conclude that the propagation speed approaches a. This conclusion is in accord with what one would expect, since for weak magnetic fields one would not expect the magnetic field to alter the sound speed appreciably. In the same limit, the slow speed will be very small compared to a, and therefore, from Eq. (II-17) we conclude that the propagation speed is equal to b_x. Thus, in this limit the slow speed approaches the intermediate speed. In other words, as the ratio of gas pressure to magnetic pressure is increased, the slow and intermediate speeds come closer together.

In the opposite limit, $a \ll b$, the fast speed approaches b, while the slow speed approaches ab_x/b. Thus, the fast propagation speed corresponds to two circles whose line of centers lies in the direction of the magnetic field, and are tangent at the origin.

A certain degree of symmetry exists in the above dispersion relation, namely that the fast and slow propagation speeds are unchanged if the magnitudes of a and b are interchanged. This follows from Eq. (II-17) since the equation is unchanged when a and b are interchanged if one remembers that b_y is the product of the magnitude of b and the sine of the angle between the magnetic field and the wave normal. This conclusion is somewhat surprising since, as we shall see below, the changes in flow properties across the waves are quite different depending upon whether a or b is larger. One should, of course, remember that as illustrated in Fig. 2, the intermediate propagation speed depends only upon b.

Let us now examine the changes in flow properties which occur across the waves. These are obtained by substituting the solutions of the dispersion relation, Eqs. (II-16) or (II-17), back into the equations for the changes, Eqs. (II-6) through (II-12). Doing this quantitatively in terms of the explicit expressions for the characteristic speeds which result from the dispersion relation is somewhat cumbersome. However, several interesting results about the wave modes can be obtained from the properties of the dispersion relation already enumerated.

Equations (II-9) and (II-11) only permit solutions with nonzero changes in v_z and B_z if the characteristic speed is equal to the intermediate speed. Since, except for very special points, the fast and slow speeds always differ from the intermediate speed, we may conclude that across fast and slow waves

$$\delta v_z = \delta B_z = 0 \qquad \text{(II-21)}$$

Thus, for these modes the magnetic field stays in the plane determined by the magnetic field ahead of the wave and the wave normal. The fast and slow waves, therefore, change only the magnitude of the tangential component of magnetic field but not its direction. On the other hand, we concluded earlier that the intermediate wave changes only the direction but not the magnitude of the tangential component of magnetic field. The normal component of magnetic field from Eq. (II-13), is of course, constant in all cases.

For the fast and slow waves there are, in general, finite changes in both v_x and v_y. These waves are therefore partially longitudinal and partially transverse. Even though we cannot separate the wave modes into purely longitudinal and purely transverse, some insight can be gained by examining the longitudinal and transverse aspects of the waves separately. The longitudinal aspects are contained primarily in the x-component of the momentum equation, Eq. (II-7). Making use of the continuity and entropy equations (II-6) and (II-12), we can rewrite this equation as

$$c^2 = \frac{\delta\left(p + \frac{B_y^2}{8\pi}\right)}{\delta\rho} = a^2 + \frac{\delta B_y^2/8\pi}{\delta\rho} \qquad \text{(II-22)}$$

The propagation speed squared is therefore the change in the longitudinal stress, resulting from both the gas pressure and the magnetic pressure divided by the change in density. This equation may also be viewed as determining the ratio of the change in B_y to the change in density. Since, by Eq. (II-20), the fast propagation speed is greater than the ordinary sound speed, it follows that the change in B_y^2 is of the same sign as the change in density for a fast wave. Conversely for a slow wave the changes are of opposite sign. Thus the propagation speed of the fast wave is greater than the ordinary sound speed, since the magnetic and gas pressures act together, while the propagation speed of the slow wave is slower than the ordinary speed, since the magnetic and gas pressures counteract each other. The fact that the gas and magnetic pressures support each other in the fast wave and counteract each other in the slow waves, is an important distinguishing feature between the two modes.

The transverse motions do not lead to as simply interpretable results. However, if we examine these by making use of the y-component of momentum conservation, Eq. (II-8), and use also Eqs. (II-10) and (II-6), we can obtain the result

$$c^2 = \frac{b_x^2}{1 - \frac{B_y}{\rho}\frac{\delta\rho}{\delta B_y}} \qquad \text{(II-23)}$$

Thus, as one might expect, if there are no changes in the longitudinal velocity which implies no change in density, the propagation speed approaches the intermediate speed which is a purely transverse wave. The departures from this speed are then given by the correction term in the denominator of the above expression. This expression is, of course, consistent with the conclusions reached earlier that for a fast wave, i. e. , a wave in which the gas and magnetic pressure changes are of the same sign, the propagation speed is greater than the intermediate speed and conversely, for a slow wave the propagation speed is less than the intermediate speed.

It is also of some interest to examine the direction of the velocity change across a wave. Using Eqs. (II-6, II-7, II-8 and II-12), we can obtain the result

$$\frac{\delta v_y}{\delta v_x} = \frac{B_x}{B_y}\left(\frac{a^2}{c^2} - 1\right) \tag{II-24}$$

As can be seen from the quadratic expressions (II-16) or (II-17), the two roots of the dispersion relation are related by the following equations

$$\begin{aligned} c_f^{\,2} + c_s^{\,2} &= a^2 + b^2 \\ c_f^{\,2}\, c_s^{\,2} &= a^2\, b_x^{\,2} \end{aligned} \tag{II-25}$$

Making use of these relations and Eq. (II-24) one can show that

$$\left(\frac{\delta v_y}{\delta v_x}\right)_f = -\left(\frac{\delta v_x}{\delta v_y}\right)_s \tag{II-26}$$

which states that the changes in velocity across the fast and the slow waves are perpendicular to each other. Remembering also that the change in velocity across the intermediate wave was in the z-direction, we reach the conclusion that the changes in velocity across the three waves are in mutually perpendicular directions.

If we imagine the waves to be produced by the motion in an arbitrary direction of a piston which forms one boundary of the gas, the relative amplitude of the three waves which are produced will depend upon the components of the piston velocity along the velocity vectors produced by each of the individual waves. An attempt to illustrate this and some of the other properties of the waves which we have derived is given in Fig. 3.

of magnetic field across an intermediate wave, the propagation speed for intermediate waves remains unchanged. Thus, the second wave will move at precisely the same speed as the first wave. We may now consider a third and fourth wave generated by the piston, and it follows from the same argument that the propagation speeds of all of these waves will be precisely equal. Since we can consider an arbitrary pulse of intermediate waves to be composed of a series of step functions, it follows that provided that the piston motion is constrained to produce only intermediate waves, the wave shape will be retained as the entire large amplitude disturbance propagates through the fluid. Thus we obtain neither steepening to form a shock wave nor spreading out as in the case of expansion fans.

The restriction on the piston motion which is required to produce a pure intermediate wave is easily seen from the condition that the change in velocity across a small amplitude intermediate wave must be perpendicular to the plane defined by the magnetic field and the wave or piston normals. Thus, the instantaneous changes in velocity or acceleration of the piston must always be perpendicular to the magnetic field at the surface of the piston.

The changes in flow properties across a large amplitude intermediate wave are obtained by summing the changes across each of the component small step function waves, which in turn are considered as differential elements. It follows immediately from Eq. (II-15) that across the large amplitude wave the changes in normal velocity, density and pressure will be zero. In evaluating the change in magnetic field we must remember that our coordinate system was chosen such that B_z was zero ahead of each small amplitude wave. Equation (II-15) therefore states that the differential change in magnetic field is in the plane of the wave front and perpendicular to the local field. Integrating a number of such changes gives the result that the magnitude of the magnetic field is unchanged across a large amplitude intermediate wave, however, the magnetic field vector can be rotated through an arbitrary large angle about an axis perpendicular to the wave front. The change in tangential velocity across the wave is from Eq. (II-15) in the direction of the change in magnetic field and is equal to $\Delta \underset{\sim}{B}/\sqrt{4\pi\rho}$. Although such a wave produces no change in the thermodynamic quantities, the normal velocity, or the magnitude of the magnetic field, it is still a large amplitude wave in the sense that the angle of rotation of the magnetic field and the change in tangential velocity can be large, i.e. of the order of radians and the propagation speed respectively.

We may anticipate that, since for small amplitude fast and slow waves the magnetic field remains in the plane defined by the wave normal and the magnetic field ahead of the wave, it will also remain in this plane for large amplitude fast and slow waves. The intermediate wave will therefore be required in flow fields in which the boundary conditions require a rotation of the plane of the magnetic field. The particular case of rotation through 180° is frequently overlooked. In this case the magnetic field appears to stay in the same plane but its tangential component changes sign. As we shall see, neither fast or slow expansion waves or shock waves can change the sign of the tangential component thus the intermediate wave will also appear in cases where such a sign change is required by the boundary conditions.

It is also instructive to look at some of the wave properties in the limits of large and small ratios of a to b. In the limit of a >> b, the fast propagation speed, as well as the changes across the fast wave, reduce to those for an ordinary sound wave. This is to be expected since in this case the magnetic pressures are too small to play any role. The fast wave, therefore, approaches a purely longitudinal wave. In the same limit, the slow wave becomes a purely transverse wave. This can be seen most easily from Eq. (II-24) which shows that the change in the y-component of velocity becomes very large as compared with the change in the x-component of velocity. It follows physically from the fact that a very small change in the longitudinal velocity produces a change in density and, therefore, a change in gas pressure which is very large compared to the magnetic pressure. Thus, only very small changes in the longitudinal velocity are required to balance the changes in the magnetic pressure. For this wave and in this limit, the fluid may therefore be considered as virtually incompressible. It then follows from Eq. (II-23) and, as we have concluded earlier, that the slow propagation speed approaches the intermediate speed.

In the opposite limit, i.e., a << b, the waves do not break up into purely longitudinal and purely transverse. In this limit, the slow wave is most easily understood. Since, in this limit, both the gas pressure and the dynamic pressure ρv^2, are small compared to the magnetic pressure, we cannot have appreciable changes in the magnetic field across the wave. Thus, the magnetic field lines will have virtually no change in direction across the wave. The plasma flow, on the other hand, is strongly coupled to these field lines. Thus, the plasma is constrained to flow in a direction parallel to the magnetic field lines. We have already observed that the propagation speed of the slow wave in this limit is equal to the sound speed multiplied by the cosine of the angle between the magnetic field and the wave propagation direction. This then corresponds to a sound wave traveling along the magnetic field lines, and therefore moving more slowly in the direction of the wave normal. The slow wave therefore becomes purely transverse for propagation perpendicualr to the magnetic field and purely longitudinal for propagation along the magnetic field. On the other hand, since the velocity change across the fast wave is perpendicular to that across the slow wave, we conclude that in this limit the fast wave is purely longitudinal for propagation perpendicular to the magnetic field, while it is purely transverse for propagation along the magnetic field.

Summary

We may summarize the major conclusions which have been reached concerning these waves as follows:

1) There are three distinct wave propagation modes which can be conveniently classified according to the magnitude of their propagation speed as fast, intermediate and slow. The velocity changes across the three waves are mutually perpendicular.

2) For fast and slow waves, both the velocity and the magnetic field remain in the plane defined by the magnetic field ahead of the wave and the wave normal. On the other hand, for the intermediate wave both the velocity and magnetic field changes are purely in the direction perpendicular to this plane.

3) For the fast mode, the magnetic pressure increases when the density increases. For the slow mode, an increase in magnetic pressure corresponds to a decrease in density. Across an intermediate wave, neither the magnetic pressure nor the density change.

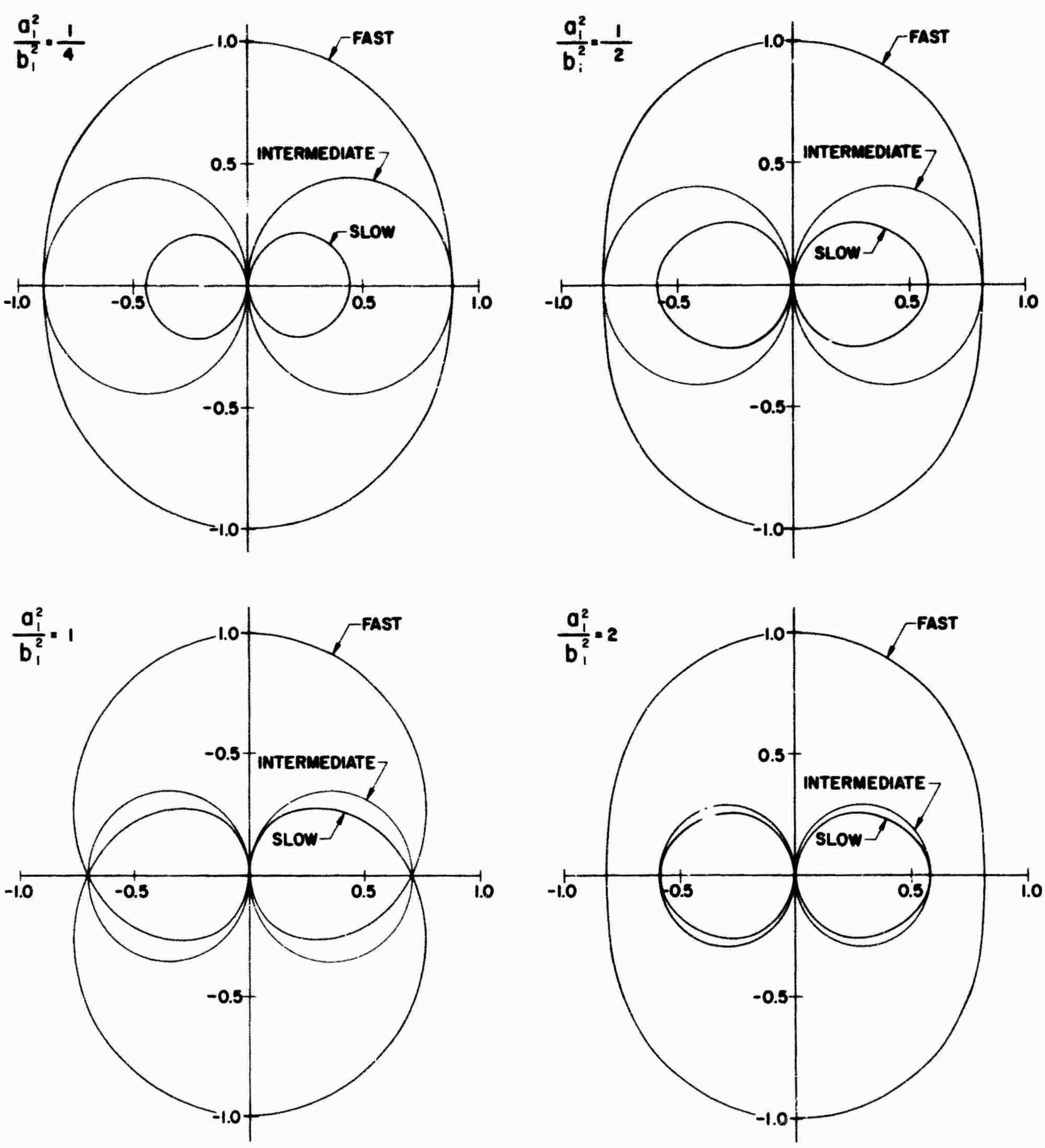

Fig. 2 Friedrichs Diagram. Polar plot showing the dependence of the propagation speeds of the three linear wave modes on the angle between the wave normal and the magnetic field. For several values of the ratio of sound speed a to Alfven speed b. Speeds have been normalized with respect to $\sqrt{a^2 + b^2}$.

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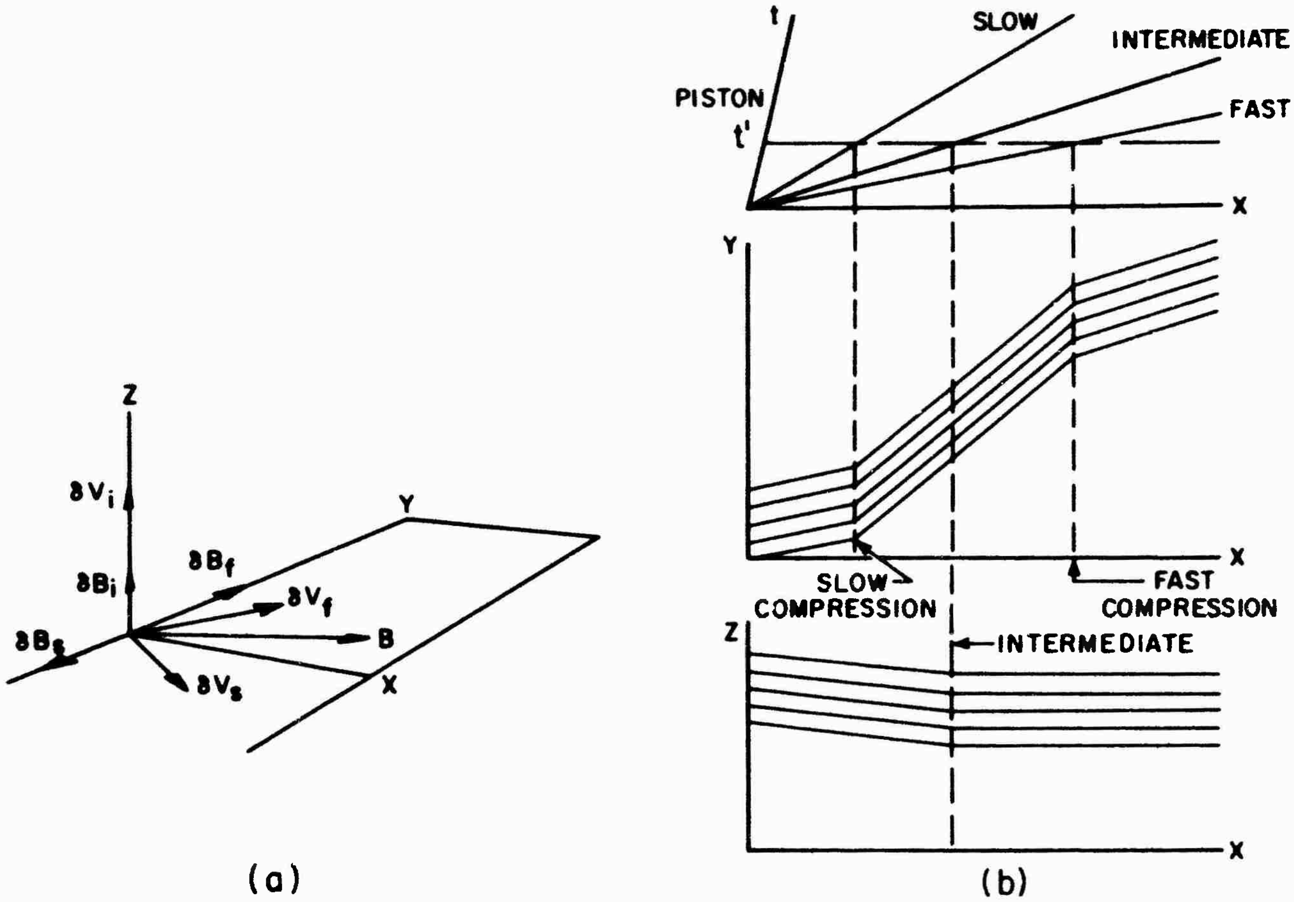

Fig. 3 Sketch of flow resulting from the instantaneous acceleration of a piston to a small velocity. In general, three waves will be emitted which separate with time as shown on the x-t diagram. The projections of the magnetic field lines on the x-y and x-z planes at a time t' are also shown for the case in which both the fast and slow waves are compressions. The changes in velocity and magnetic field across the three waves are illustrated in the vector diagrams. The initial magnetic field, δv_s and δv_f are in the x-y plane. δv_f must lie within the acute angle between the magnetic field and the y-axis. The three velocity changes are mutually perpendicular. The signs of δB_s and δB_f were also chosen for compression waves.

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SECTION III

LARGE AMPLITUDE ISENTROPIC WAVES AND SHOCK FORMATION

The solution to a nonlinear flow problem can be built up by considering it as a series of small amplitude waves, each propagating through a medium which has been modified by previous waves. In this manner, it is possible to discuss problems with arbitrarily large amplitudes. The concept of a large number of isentropic small amplitude waves describing the flow breaks down in the case where shock waves are formed. However, the nonlinear isentropic solutions can be used to predict when shock waves occur. The shock waves themselves will be discussed in the next section. In this section, we will consider the nonlinear waves related to each of the linear wave propagation modes. We will consider only the case in which the waves are all propagating in one direction, i.e., as though they were generated at the boundary of a semi-infinite plasma. For the special case in which the boundary condition is changed suddenly, the fact that the three propagation speeds are different separates the resulting nonlinear waves. Thus, for this case the nonlinear description of the individual modes can be used to obtain a general solution for an arbitrary instantaneous change in the boundary condition. The more general case in which several wave modes exist at the same place or waves of the same mode exist in the same place propagating in opposite directions will not be considered. Problems of this kind can also be treated by a generalization of the procedures to be described; however, in most cases, they involve considerable labor.

We shall show that compression waves for both the fast and the slow modes tend to steepen to form shock waves, whereas the expansion waves for these two modes tend to spread out with time so that the gradients become less steep. The intermediate wave, on the other hand, has the rather surprising property that even for large amplitudes, it remains a linear wave. Thus, even for large amplitude, an intermediate wave of arbitrary shape will retain its shape as it propagates through the medium.

Intermediate Large Amplitude Waves

Let us imagine a semi-finite uniform plasma bounded by a piston. At time zero the piston is moved such as to produce a step function small amplitude intermediate wave. A short time later the medium will still be undisturbed ahead of the region to which the wave has propagated, i.e., for distances greater than $b_x t$ from the piston. In the region between the piston and the instantaneous location of the wave, the medium will again be uniform, but at a slightly different condition than the condition existing ahead of the wave. If at this time the piston velocity is again changed instantaneously so as to produce a second intermediate wave, we may examine the propagation speed of this second wave. In order to do this we must determine the conditions behind the first wave. Since, as we concluded in the previous section, there is no change in density, normal component of velocity, or normal component

Fast and Slow Waves

Let us now consider the case in which the piston motions are such as to generate two successive small amplitude waves which are either both fast or both slow waves. Differences in the propagation speeds of the two waves result from two causes. First, there is a change in the propagation speed of the wave relative to the fluid due to the changes in magnetic field and density across the wave. Secondly, there is a change in fluid velocity across the first wave, which means that the second wave is riding on a fluid which is already moving. If the first wave is a compression wave, the fluid behind it is moving at a slow velocity in the direction of propagation of the wave. Thus, the second wave, even if it had the same propagation speed relative to the fluid, would tend to catch up with the first wave. In most cases, we will find that the change in fluid velocity is the predominant effect.

A quantitative determination of the conditions under which the two waves will catch up with each other can be obtained by examining the expression $\frac{\rho}{c} \frac{\delta(v_x + c)}{\delta\rho}$. $\delta(v_x + c)$ is the propagation speed of the second wave relative to the first. If this is of the same sign as the propagation speed of the first wave, the second wave will overtake the first. Thus, $\delta(v_x + c)/c$ positive implies steepening. Therefore, if the above expression is positive steepening occurs for the positive $\delta\rho$, i.e. compression waves. If it were negative, steepening would occur for rarefaction waves. Making use of the jump conditions across the wave and the dispersion relation, this quantity can, with some algebraic manipulation, be written as

$$\frac{\rho}{c} \frac{\delta(v_x + c)}{\delta\rho} = 1 + \frac{1}{2} \frac{(\gamma - 1)\, a^2 b_y^{\,2} + (c^2 - a^2)^2}{a^2 b_y^{\,2} + (c^2 - a^2)^2} \qquad \text{(III-1)}$$

It can easily be seen that this expression is always positive. It is interesting to note that this quantity which describes the rate at which waves catch up with one another is insensitive to the gas conditions. It is always between $\frac{\gamma + 1}{2}$ and 3/2.

If we now extend this argument to a smooth pressure pulse which we consider to be made up of a large number of small amplitude waves, we see that for both the fast and the slow mode, the compression parts of the pulse tend to steepen to form shock waves, while the rarefaction portions tend to separate. Thus in terms of the illustration given in general terms in Fig. 1 (c), we may now interpret the pulse shape which is shown as the density or pressure profile. The maximum density point overtakes the lower density regions ahead of it while it moves away from the ones behind. As mentioned earlier, across the discontinuities which are formed we can no longer assume that the gradients are too small for viscous dissipation and Joule heating to be important. Thus, there will be an entropy change and we must consider such shock waves separately. For rarefaction waves on the other hand, the lines of constant conditions spread apart and thus the gradients become less steep as time progresses. The isentropic theory is therefore applicable to all rarefaction waves and to compression waves provided that they have not yet steepened to form shocks.

Nonlinear waves consisting only of one wave mode propagating in only one direction are generally referred to as simple waves. In a simple wave, plasma conditions are constant along lines moving at the propagation speed of the individual small amplitude waves, $v_x + c$. (Such lines do not cross any waves.) Thus, if conditions are known along one boundary for example, a piston which we imagine to be producing the flow, then conditions at later time can be determined by projecting constant properties along these lines. Since the relative changes in the flow properties across the component small amplitude waves are given by Eqs. (II-6) through (II-13), integration of these equations allows one to express all of the flow properties in terms of one of them. Thus, the changes which can occur across simple waves can be determined independently of the piston motion. The rate of change of velocity of the piston will determine at what point in the flow field these changes in flow properties actually occur.

Although some of these relations can be integrated formally, a somewhat more surveyable graphical representation of the integrals was suggested by Shercliff.[3] For the case of one-dimensional time-dependent flows, (B_x constant), the differential equation relating B_y and ρ contains only these two variables. This can easily be seen by eliminating c^2 from Eqs. (II-22) and (II-23) and remembering that sound speed is related to the density by the isentropic law. Introducing nondimensional variables

$$B_y' = \frac{B_y}{B_x} \qquad \rho' = \left(\frac{a}{b_x}\right)^{\frac{2}{\gamma}} = \left(\frac{4\pi\gamma}{B_x^2}\,\frac{p}{\rho^\gamma}\right)^{\frac{1}{\gamma}}\rho = \frac{\rho}{\rho_o} \qquad \text{(III-2)}$$

the resulting differential equation can be written in the form

$$\left(\frac{d B_y'}{d\rho'}\right)^2 - \left(\frac{1 + B_y'^2 - \rho'^\gamma}{\rho' B_y'}\right)\frac{dB_y'}{d\rho'} - \rho'^{(\gamma-2)} = 0 \qquad \text{(III-3)}$$

The variable ρ' is indeed proportional to the first power of the density since from the isentropic law ρ_0 is a constant. ρ_0 may be regarded as the density to which the gas must be expanded or compressed isentropically in order to reach the condition $a = b_x$. The above equation is quadratic in $\frac{dB_y'}{d\rho'}$ and therefore may be factored into two first order differential equations. These two equations correspond to the trajectories in the $B_y' - \rho'$ plane across fast and slow waves respectively. The numerical solutions of these equations for the case $\gamma = 5/3$ are shown in Fig. 4(a). As can be seen, these solutions correspond to a one-parameter family of curves. The parameter could be expressed as the value of ρ' which occurs at $B_y' = 0$. An arbitrary initial gas condition determines a point in this plane. Through each point there are two lines corresponding to the trajectories along fast and slow simple waves. The changes in the remaining flow properties along these trajectories can be obtained by first solving the dispersion relation Eq. (II-16) for the speed of propagation, c, in terms of B_y' and ρ'. Then, integrating Eqs. (II-6) and (II-8), the changes in the x and y components of velocity can be determined. These results are shown in Figs. 4(b), (c), and (d) with the velocities nondimensionalized with respect to the velocity $B_x/\sqrt{4\pi\rho_0}$. The equations determining the change in velocity do not depend upon the magnitude of the velocity. Thus, the integration gives only the change in velocity between two points along a trajectory and not the absolute magnitude. In Figs. 4(c) and (d), the velocity coordinates are to be regarded as the change in that component of velocity which would occur between a given point on the trajectory and the point at which B_y' is equal to zero.

Figure 4 contains all of the information required to determine conditions across fast and slow simple waves. If, for example, the variation of density with time is known at a fixed position, then the variation in B_y, v_x, v_y and c can be determined from Fig. 4. Conditions in the remainder of the flow field are then known to be constant along lines moving at a velocity of $c + v_x$. Thus, conditions in the whole flow field are determined.

Across large amplitude fast and slow waves the change in B_z and v_z is zero since it is zero across the small amplitude waves. Remembering that our choice of the y and z directions was such as to make B_z zero, it is appropriate to consider the B_y' axis in Fig. 4(a) as representing the tangential component of magnetic field. Thus if we chose our coordinate relative to a fixed direction in the plane of the wave front and defined the tangential magnetic field in terms of its magnitude and the angle $\phi = \tan^{-1} \frac{B_z}{B_y}$, the trajectories across fast and slow waves are confined to planes of constant ϕ and have the shapes shown in Fig. 4 (a). As mentioned earlier in order to go from one value of ϕ to another, the intermediate wave is required. From the properties of the intermediate wave enumerated above, the trajectories of intermediate waves on a three-dimensional extension of Fig. 4(a) correspond to the lines obtained by taking any point on the plane shown and rotating it about the ρ' axis.

Several other features of the diagram shown in Fig. 4(a) are worth commenting on. First, as was apparent from the jump conditions across individual waves or, what is equivalent, the differential equations which determined these curves, for fast waves the density always increases as the magnitude of the magnetic field increases while for slow waves the density decreases as the magnitude of the magnetic field increases. Expansion to a vacuum, $\rho = 0$, can only be achieved by means of a slow expansion wave. If we followed a fast trajectory toward decreasing densities we reach $B_y = 0$ at a finite value of ρ. There is one exception to this which is somewhat disguised in our diagram by the nondimensionalization of the density. For the case in which B_x is equal to zero, it is possible to have a fast expansion go to zero density. It is also apparent from the diagram that $B_y = 0$ can only be reached by either a fast expansion or a slow compression. Conversely it is only possible to go away from $B_y = 0$ with either a fast compression or a slow expansion. Furthermore, for a fast expansion, the point $B_y = 0$ is always reached at a value of ρ' which is less than one. And conversely, the slow isentropic compressions would only reach $B_y' = 0$ at ρ' greater than one. This latter relation turns out also to be valid across the corresponding slow shock waves.

In our discussion we have restricted ourselves to a considerable extent to the case of one-dimensional time-dependent flows. Simple waves also exist in steady two-dimensional flows in which the flow velocity is larger than the wave propagation speed. Such flows will not be discussed in detail here except to point out that the major conceptual difference between the two cases lies in determining the trajectories of the elementary small amplitude waves. In order to achieve a steady flow situation the waves must propagate relative to the fluid at the same speed at which they are blown back by the fluid velocity. Thus, the angle which the wave normal makes with respect to the flow velocity is determined by the conditions that the normal component of velocity be equal to the propagation speed, c, of the wave. Several references on two-dimensional flows are included in the bibliography.

Instantaneous Piston Acceleration

Two possible flow patterns which can occur when the boundary conditions are changed instantaneously to a different constant value are illustrated in Fig. 5. In this case all of the waves are generated at the boundary at the time at which conditions there are changed. Thus, flow properties are constant along lines in the x-t plane emanating from this point. Since the propagation speeds of the three wave modes are different, they will separate as time progresses. Thus, the entire flow can be described as consisting of three simple waves corresponding to the three modes. Since, by definition, the fast mode has the fastest propagation speed, a fluid element some distance from the boundary will first experience a change corresponding to a fast simple expansion fan then an intermediate and finally a slow one. The regions occupied by the fast and slow expansion fans widen as time progresses since the propagation speed at the back edge of a rarefaction wave is slower than that at the front edge. Since in general, there is a finite difference between the propagation speeds of the different wave modes, there will be regions between the different waves which also spread with time in which conditions are completely uniform. Since the propagation speed of the intermediate wave does not change as the fluid moves through the wave, the intermediate wave will remain a discontinuity.

In case the boundary conditions are such that one of the waves corresponds to a compression wave a shock will be formed. For the case of instantaneous change of the boundary condition the shock will move at constant velocity and thus, conditions behind it will be independent of time. The jump conditions across shock waves will be discussed in the next section. If the other wave mode is a rarefaction wave the techniques of this section can be applied to calculate the changes and flow property across the expansion wave. A case of this type is sketched in Fig. 5(b). We will return to a discussion of this diagram in Section VI when we consider the application of shock waves to the production of high temperature plasma samples for laboratory study.

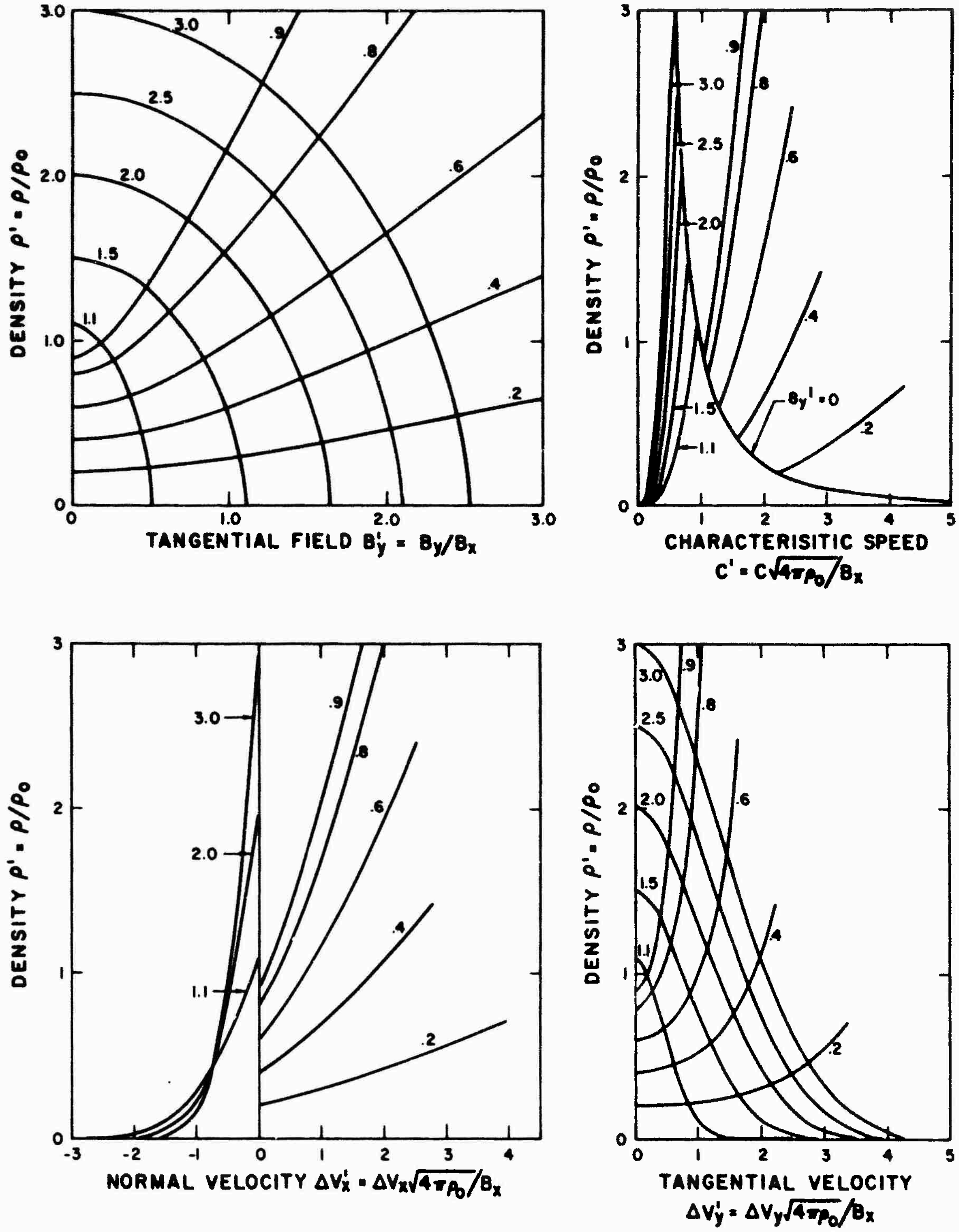

Fig. 4 Changes in fluid properties across one-dimensional time-dependent fast and slow simple waves for $\gamma = 5/3$. Curves are labeled according to the value of ρ' at $B'_y = 0$, for fast waves this is less than one and for slow waves it is greater than one. Both components of flow velocity have been taken to be zero at $B'_y = 0$, thus, these curves should be considered as the change in velocity from this point on the trajectory. Only the fast characteristic speed is given along a fast wave and only the slow characteristic along a slow wave.

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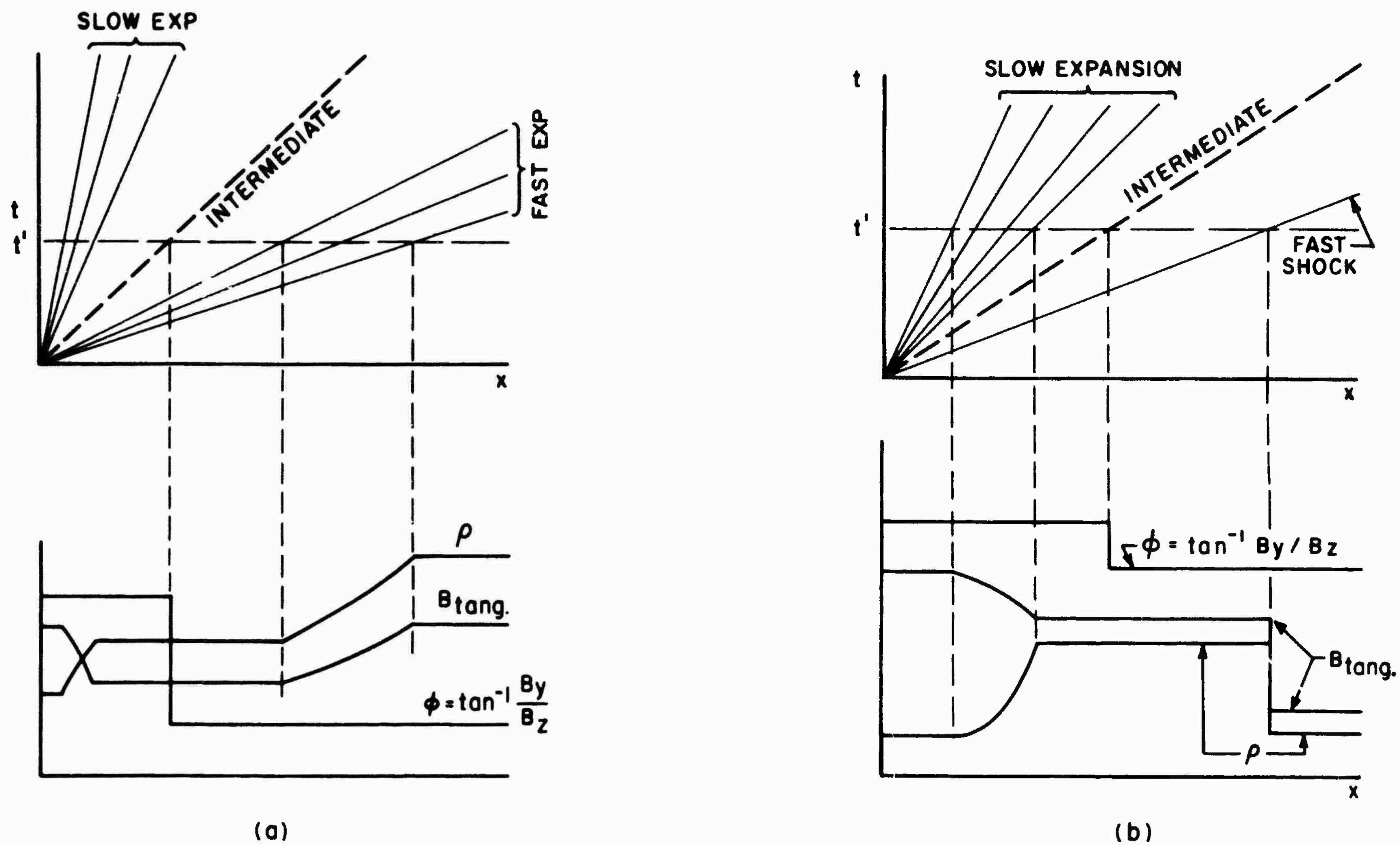

Fig. 5 Qualitative description of flow patterns resulting from an instantaneous change in boundary conditions to a new constant condition. The lower curve shows fluid properties as a function of position for a particular time t'. In a) both the fast and slow waves are expansions and the flow properties follow the trajectories given in Fig. 4. Note that there are four separate regions within which the flow is completely uniform. In b) the fast wave is a compression, thus, the changes across it occur suddenly and are obtained from the shock relations discussed in Section V.

SECTION IV

SHOCK WAVES

From our previous discussion compression waves of either the fast or the slow mode steepen to form discontinuities. We may therefore expect to find two different types of shock waves depending upon which mode of small amplitude waves formed the shock.

The changes in flow properties which occur across shock waves may be determined without reference to the specific dissipation processes occurring within the shock wave. The dissipation processes are related only to the structure within the shock wave which will be discussed briefly in Section V. For the present discussion, we will consider the thickness of the shock wave to be infinitessimally small compared to the scale of the flow field.

Let us go to a coordinate system moving with the instantaneous shock velocity and draw two planes parallel to the plane of the shock, one on either side of the shock wave. Since the shock wave itself is very thin, these two planes can be very close together in terms of the scale of the overall flow field. For a steady flow there can be no net rate of accumulation of either mass, momentum, energy, or magnetic field in the region between the two planes. Thus if a shock wave is to exist between these two planes, the conditions at the planes must satisfy the restrictions that the fluxes of the above quantities are the same across both planes. The relations equating these two fluxes then determine the jump conditions which are allowed across shock waves. If the planes are very close together the time that it takes the fluid to go from one plane to the other is very small compared to the time scale on which the overall flow field changes. Thus, even for an unsteady flow, we may consider this small portion of the flow as being steady. The resulting shock relations can therefore be used in unsteady flows.

Let us denote the two planes by the subscripts 1 and 2. Further, let us choose a coordinate system such that the x-direction is the normal to the wave front and let us assume that the coordinate system is moving at a velocity such that $v_{y1} = v_{z1} = 0$ and again orient our coordinate system such that $B_{z1} = 0$. The equations which we will obtain are in essence the nonlinear versions of the jump conditions across small amplitude waves which we discussed earlier. One exception to this is that entropy is not conserved. This equation must be replaced by the conservation of energy flux.

Let us consider first the conservation of flux of magnetic field. The statement that no magnetic field is accumulated between the two planes is equivalent to the statement that for a steady state $\underset{\sim}{\nabla} \times \underset{\sim}{E} = 0$. Thus, the tangential components of electric field, E_y and E_z, must be the same on both sides of the shock front. Since our two control planes are outside of the shock, the current density is small at these points and we may use the relation

$$\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c_e} = 0 \tag{IV-1}$$

where c_e is the velocity of light. Making use of this equation and the fact that E_y and E_z do not change across the shock, we obtain the equations

$$v_{x2} B_{y2} - v_{y2} B_x = v_{x1} B_{y1} \tag{IV-2}$$

$$v_{x2} B_{z2} - v_{z2} B_x = 0 \tag{IV-3}$$

Note that no subscript is required on B_x since it must be the same on both sides of the shock.

The requirement of the conservation of mass leads to the relation

$$\rho_2 v_{x2} = \rho_1 v_{x1} \tag{IV-4}$$

The momentum flux across a plane is composed of the flow of fluid momentum across the plane, (i e., the product of mass flow and velocity) the gas pressure and the magnetic stresses acting on the plane. Thus, we obtain for the three components of the momentum equation

$$\rho_2 v_{x2}^2 + p_2 + \frac{B_{y2}^2}{8\pi} = \rho_1 v_{x1}^2 + p_1 + \frac{B_{y1}^2}{8\pi} \tag{IV-5}$$

$$\rho_2 v_{x2} v_{y2} - \frac{B_x B_{y2}}{4\pi} = - \frac{B_x B_{y1}}{4\pi} \tag{IV-6}$$

$$\rho_2 v_{x2} v_{z2} - \frac{B_x B_{z2}}{4\pi} = 0 \tag{IV-7}$$

Finally, the energy equation consists of the flow of thermal and kinetic energy of the fluid as well as the flow of electromagnetic energy which is given by the Poynting vector. Making use of Eq. (IV-1) we obtain the relation

$$\rho_2 v_{x2}\left(\frac{\gamma}{\gamma-1}\ \frac{p_2}{\rho_2} + \frac{1}{2} v_{x2}^{\ 2} + \frac{1}{2} v_{y2}^{\ 2}\right) + \frac{B_{y2}}{4\pi}\left(B_{y2} v_{x2} - B_x v_{y2}\right)$$

$$= \rho_1 v_{x1}\left(\frac{\gamma}{\gamma-1}\ \frac{p_1}{\rho_1} + \frac{1}{2} v_{x1}^{\ 2}\right) + \frac{B_{y1}^{\ 2} v_{x1}}{4\pi} \qquad \text{(IV-8)}$$

The above equations are sufficient so that if all of the conditions on one side of the shock are known, the conditions on the other side are determined. Note that conditions ahead of the shock include the flow velocity relative to the shock. Thus the shock speed as well as the thermodynamic and magnetic field conditions must be specified.

The solution of these equations results largely from straightforward, but somewhat tedious algebraic manipulation. In the discussion which follows we will first describe, but not necessarily prove, some of the general results which apply to the jump conditions across shock waves. Following this, we will discuss graphs of the jump conditions for various special cases. These graphs have been chosen to give the reader some feeling for the changes which can occur across shock waves. They are not intended to provide a complete summary of the shock relations for computational purposes.

Shock waves are always compression waves. This conclusion is to be expected from the shock formation arguments given in the previous section which showed that compression waves tended to steepen to form discontinuities while rarefaction waves spread apart. It does not follow immediately from the equations written above since the sign of the velocity may be changed throughout without changing the equations. Thus, solutions of the above equations could correspond to the flow going either from low to high density or from high density to low density. If, however, one examines the entropy on the two sides of the shock it can be shown that the entropy is always higher on the higher density side.[4] Since the entropy must increase with time, it follows that the flow must go from the low density side toward the high density side.

The physically realizable solutions of these equations can be divided into two categories which have been named fast and slow shocks. The limit of weak fast and slow shock waves are the fast and slow small amplitude disturbances discussed in Section II. The magnetic field changes across these shock waves are qualitatively the same as they would be for fast and slow small amplitude compression waves. The tangential component of magnetic field increases across fast shocks and decreases across slow shocks. Furthermore, as will be shown below, the magnetic field behind the shock is in the plane defined by the wave normal and the magnetic field ahead of the shock.

The fast disturbance speed ahead of a fast shock is always less than the normal component of flow velocity, while the fast disturbance speed corresponding to conditions behind the shock is always greater than the normal component of the flow velocity behind the shock. The corresponding statement can also be made for slow shocks relative to the slow disturbance speed. The proof of this for weak shock waves follows directly from the arguments used to show that compression waves steepen. Consider one of the waves used in the steepening argument to be a weak shock. It was then shown that a small amplitude wave behind this wave would overtake it, thus, showing that the small amplitude disturbance speed behind the waves was greater than the flow velocity. Correspondingly the wave under consideration would overtake a small amplitude wave ahead of it thus, showing that its velocity relative to the fluid was larger than the propagation speed ahead of it. A plausibility argument for this statement for stronger shock waves can also be given on the basis of the shock steepening analysis. If we imagine a shock to be formed from the steepening of a gradual pressure pulse, then as the first few sound waves cross, the shock will gradually become stronger. Each increase in shock strength is then directly related to a small amplitude wave overtaking the shock. Thus we would not expect to be able to produce shock waves in this manner for which small amplitude disturbances from behind cannot catch up. If the flow velocity ahead of the shock were less than the disturbance speed then a wave coming from behind will not pile up at the shock, but would go out ahead. Thus, we also require that the flow velocity ahead be greater than the disturbance speed. Direct verification of this restriction on the velocities ahead and behind shock waves can be obtained by examining the algebraic solutions of the shock relations. A further restriction on the flow speeds which will be discussed in more detail below is that for fast shocks the flow velocity must be greater than the intermediate speed on both sides of the shock while for slow shock waves the flow velocity must be less than the intermediate disturbance speed on both sides of the shock.

The proof of the fact that the magnetic field behind the shock must lie in the plane defined by the magnetic field ahead and the wave normal proceeds as follows. Eliminating v_{y2} between Eqs. (IV-2 and IV-6) and eliminating v_{z2} between Eqs. (IV-3 and IV-7) we obtain the relations

$$\left(\frac{v_{x2}^2}{b_{x2}^2} - 1\right) B_{y2} = \left(\frac{v_{x1}^2}{b_{x1}^2} - 1\right) B_{y1} \qquad \text{(IV-9)}$$

$$\left(\frac{v_{x2}^2}{b_{x2}^2} - 1\right) B_{z2} = 0 \qquad \text{(IV-10)}$$

It follows from Eq. (IV-10) that either the z-component of magnetic field behind the wave is zero or the flow velocity must be equal to intermediate propagation speed. The latter case requires from Eq. (IV-9) that either the flow velocity ahead be also equal to the intermediate speed ahead of the wave or that the y-component of magnetic field ahead of the wave be zero. If B_{y1} is zero then the plane of the magnetic field ahead of the wave is not defined and it does not make sense to distinguish between the y and z components of the magnetic field behind the wave. If the flow velocity is equal to the intermediate speed both ahead and behind the wave, this simply corresponds to the large amplitude intermediate wave discussed in the previous section. Since intermediate waves do not steepen to form discontinuities and since there is no entropy change across them it does not seem appropriate to refer to them as shock waves. It should be remembered, however, that an almost discontinuous rotation of the plane of the magnetic field can occur across such a wave, if it is initiated sufficiently rapidly. Thus, excluding the discontinuous intermediate wave, a finite value of B_{z2} cannot occur behind a shock wave. Thus, the magnetic field and the velocity are entirely in the x-y plane on both sides of the shock front and Eq. (IV-3) and (IV-7) need not be considered in further discussion of the conservation equations.

We shall now show that shock waves with velocities greater than the intermediate speed ahead and less than the intermediate speed behind cannot occur. As can be seen from Eq. (IV-9) this statement is completely equivalent to the statement that the sign of the tangential component of magnetic field cannot change across a shock wave. There has been considerable confusion on this point in the literature since solutions of the conservation Eqs. (IV-2) through (IV-8) which violate the above condition do exist. These solutions, however, cannot occur in nature and should therefore be regarded as extraneous.* A portion of the confusion is related to the fact that a number of authors who have discussed solutions of the conservation equations did not recognize that some of these solutions were extraneous and discussed them at considerable length. Since the extraneous solutions are considerably more complicated than the real ones this leaves the overall impression that the solution of the shock equations is much more complex than it actually is. Further confusion resulted from the fact that these extraneous solutions were originally referred to as unstable shock waves. It was recognized that this was not an ordinary instability growing exponentially with time, but rather that it corresponded to a sudden disentegration of the shock wave. Numerous papers have been written on the waves produced when such a shock wave disintegrated. In the present discussion we shall use essentially the same arguments which were initially proposed as leading to the instability of these solutions. The logical conclusion of the argument as presented here is, however, that these solutions cannot occur in nature even for very short times and thus that they are extraneous solutions of the mathematics which do not correspond to physical reality.

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*In the literature these solutions are usually referred to as unstable or non-evolutionary shock waves. We prefer the term extraneous since it implies more directly that these solutions cannot occur in nature even for very short times.

Let us imagine that a flow containing an extraneous shock does exist and focus our attention on a small region of this flow in the neighborhood of the shock wave. This small region may be regarded as having been produced by the instantaneous acceleration of a piston to a velocity corresponding to conditions behind the shock wave. Since for all solutions of the conservation equations B_{z2} and v_{z2} are zero the required piston motion is purely in the x-y plane. If we not consider a second case in which the boundary conditions are changed slightly so as to require an arbitrarily small z component of velocity at the piston, we will find that no neighboring solutions of the flow problem exist. The only wave mode which can produce the required z component of velocity or magnetic field is the intermediate wave. We would therefore expect that a small amplitude intermediate wave would also be emitted from the piston at the time at which it is accelerated. There is, however, no place in the flow field where the intermediate wave can exist. The velocity of the extraneous shock relative to the fluid ahead of it is greater than the intermediate propagation speed. Thus, the intermediate wave cannot propagate ahead of the shock wave. On the other hand, the velocity of the extraneous shock wave relative to the fluid behind it is less than the intermediate propagation speed, thus the intermediate wave cannot remain behind the shock wave. Furthermore, since the plane of the magnetic field cannot rotate across the shock wave the intermediate wave can also not exist in the middle of the shock wave. Thus a flow containing an extraneous shock does not have any neighboring solutions corresponding to a small change in the boundary conditions which requires an arbitrarily small angle of rotation of the plane of the magnetic field.

This difficulty does not occur for the allowed fast or slow shocks. For the fast shock the flow velocity is greater than the intermediate propagation speed on both sides of the shock and thus the required intermediate wave could exist behind the shock wave. Correspondingly for the allowed slow shock solutions the flow velocity is less than the intermediate speed on both sides of the shock and thus the intermediate wave could exist ahead of the shock wave. As has already been observed across the extraneous shock solutions the tangential component of magnetic field changes sign, i. e., rotates through 180^o. We shall now show that boundary conditions which require such a 180^o rotation of the tangential component of magnetic field can always be satisfied by flows which contain allowed fast and slow waves and a 180^o intermediate wave. Thus there is no consistent set of boundary conditions which would require the existence of an extraneous shock in the solution. One might argue that for the case in which the rotation in the intermediate wave is precisely 180^o, the flow solution is not unique. However, since the lack of uniqueness exists only over an immeasurably small range of angles one would not expect to be able to produce an extraneous shock in any physical situations.

In order to show that an alternate solution always exists let us consider again a piston on which boundary conditions are changed instantaneously. For the present discussion, it will be more convenient to consider the piston as an insulator. In this case, the tangential component of flow velocity at the surface is not necessarily the velocity of the piston. Thus, the y and z components of velocity cannot be controlled directly by the piston motion.

However, since the piston is an insulator it is possible to control the magnetic field at the surface. The appropriate boundary conditions are therefore the x component of velocity and the y and z components of magnetic field at the surface. A one to one correspondence exists between flows specified by the above boundary conditions and the flows resulting from conducting piston boundary conditions which are the specification of the three components of velocity. Let us consider first, the set of cases in which B_z is zero at the piston and B_y is of the same sign as it is before the boundary conditions are changed. Solutions must be possible for arbitrary values of v_x and arbitrary positive values of B_y. Furthermore, since the sign of B_y is unchanged these solutions do not contain either an intermediate wave or an extraneous one. The case of two extraneous shocks which would change the sign of B_y twice is also not possible since they would overtake one another. The solution in this case must therefore consist only of the allowed waves. We now change the boundary conditions such that the magnitude of the tangential component of B at the piston remains the same, but its direction is changed. Then the flow solution remains identical except that an intermediate wave which rotates through the appropriate angle must be inserted between the fast and slow waves. This follows from the fact that across the intermediate wave the magnitude of the magnetic field, the normal component of the flow velocity and the density and pressure are all unchanged. Thus, the quantities which are relevant for determining the changes across the fast and slow waves are unchanged by the presence of the intermediate wave. In terms of these quantities the flow solution for an arbitrary angle of rotation is known once it is known for a particular angle of rotation. The magnitude of the tangential component of velocity is not given quite as directly since it can change across an intermediate wave. We can, however, conclude that, given a flow solution corresponding to no rotation of the magnetic field, solutions also exist for all other angles of rotation of the magnetic field. In particular the case of 180° rotation is included. Thus boundary conditions which might suggest the extraneous shock can also be satisfied by a solution of the type just described.

In the process of eliminating the extraneous solutions from consideration we have derived two restrictions on the allowed shock solutions. These may be conveniently summarized as the statement that for fast shocks the flow velocity on both sides of the shock must be greater than the intermediate propagation speed while for slow shocks the flow velocity on both sides must be less than the intermediate propagation speed. Using Eq. (IV-9) this statement also shows that the sign of the tangential component of magnetic field is unchanged across either a fast or a slow shock wave. It also follows immediately from the velocity restriction that a slow shock cannot overtake a fast shock while a fast shock necessarily overtakes a slow shock.

Fast Shocks

Let us now turn to a more quantitative description of the solution of the shock equations. The conditions behind a shock are specified if conditions ahead including the flow velocity relative to the shock are known. Even in non-dimensional terms there are four parameters which are required to define a particular shock a_1/b_1, M_1, θ_1, and γ. The subscript 1 will be used to define conditions in the low density stream ahead of the shock. a_1/b_1 may be regarded as defining the ratio of gas to magnetic pressure ahead of the wave. The Mach number M_1 specifies the shock strength and is defined as the flow velocity in shock coordinates divided by the appropriate (fast for fast shocks and slow for slow shocks) small amplitude disturbance speed. $\theta_1 = \tan^{-1} B_{y1}/B_x$ defines the angle of the magnetic field relative to the shock normal. The thermal properties of the gas can be defined in terms of the ratio of specific heats γ. A complete description of the shock relations for the entire range of these parameters clearly requires numerous graphs. We shall restrict ourselves to considering only $\gamma = 5/3$ and shall only present graphs for selected values of other parameters. The intent is to give the reader some impression of the significance of these parameters in determining conditions behind the shock rather than to present a complete survey suitable for use in computation of flow fields.

In Fig. 6 the conditions behind fast shocks propagating perpendicular to the magnetic field ($B_x = 0$) are plotted with a_1/b_1 as a parameter. The density ratio across the shock is relatively insensitive to a_1/b_1. For weak shocks, the density ratio must, of course, approach unity. For strong shocks, all of the curves approach the limiting density ratio of four for the case of $\gamma = 5/3$. In the strong shock limit, the curves must approach one another since the flow kinetic energy is large compared to either the thermal or the magnetic energy, and thus, the ratio becomes insignificant. The temperature change across the shock has been normalized with respect to the flow kinetic energy ahead of the shock wave. In the limit of strong shock waves, virtually all (15/16) of the flow energy ahead of the shock becomes converted to thermal energy behind. In the limit of weak shocks the temperature change must of course approach zero. It is interesting to notice that for intermediate strength shocks, the temperature change is significantly smaller for the case of a strong magnetic field ahead of the shock ($a_1/b_1 < 1$) than it is for the case of weak magnetic fields ahead of the shock. This difference is accounted for by the fact that the magnetic energy density behind the shock is higher than it is ahead of the shock, thus, in the case of strong magnetic fields, some of the initial kinetic energy is converted into magnetic energy and a smaller amount remains for thermal energy. It is a fairly general property of fast shock waves that the presence of the magnetic field tends to reduce the temperature change because of the energy which must go into the magnetic field.

Figure 7 shows conditions behind fast shocks propagating along the magnetic field ($B_{y1} = 0$). Since ordinary hydrodynamic shock waves produce no transverse velocity one might expect that in this situation one would obtain only the ordinary hydrodynamic solutions. This is, however, not necessarily the case. If the magnetic field is sufficiently large, the ordinary shock solution may give a flow velocity behind the shock which is less than the intermediate propagation speed. In terms of our previous discussion, this would correspond to an extraneous solution and is thus not allowed. Under these conditions, the allowed fast shock solution has a tangential component of magnetic field behind the shock although it is zero ahead of the shock. Such shocks are referred to as switch-on shocks, since the tangential field is switched on by the shock. In this case, it follows from Eq. (IV-9) that the flow speed behind the shock is precisely equal to the intermediate speed. This conclusion leads to an apparent paradox, since it would seem to be perfectly reasonable to accelerate a piston along magnetic field lines to a velocity which corresponds to such an extraneous ordinary shock solution. In this case one might expect to find the ordinary shock rather than the switch-on shock. This paradox can be resolved by the fact that there is also a slow switch-off shock wave. For a switch-off shock wave, the tangential component of magnetic field is finite ahead of the shock and zero behind it. Again from Eq. (IV-9), the velocity ahead of a switch-off shock wave is equal to the intermediate speed ahead of the shock. Thus, for the particular case just given, the switch-on and switch-off shock waves would travel at precisely the same speed. If both are created at the same instant of time, the net result of a switch-on shock followed immediately by a switch-off shock would be indistinguishable from the ordinary hydrodynamic shock solution. Although the distinction between a composite shock, made up of a switch-on and switch-off shock, and an ordinary hydrodynamic shock seems somewhat artificial for the case in which the shock waves are propagating precisely along the magnetic field lines, the distinction does have some significance if the wave propagation is at a slight angle to the magnetic field. In the latter case, the fast shock will be not quite a switch-on shock and thus, the flow velocity behind the shock will be slightly greater than the intermediate speed behind the shock. Similarly, the propagation speed of the almost switch-off shock will be slightly less than the intermediate propagation speed. Thus, the slow shock will move slightly more slowly than the fast shock and as time progresses, the two shock waves will separate.

For the case $a_1/b_1 = 0$, the ordinary shock solutions do apply for Mach numbers greater than two in the case $\gamma = 5/3$. However, in the range of Mach numbers between 1 and 2, the appropriate fast shock solutions are switch-on shock waves. For $a_1/b_1 = 1$, the ordinary shock solutions are possible down to Mach number 1 without violating the condition of not crossing the intermediate speed. Thus, no switch-on shock waves exist for $a_1 > b_1$. For all of the curves on Fig. 7 with $a_1/b_1 < 1$, the ordinary shock solution applies above some critical Mach number which lies between 1 and 2. Note that the Mach number which is used as abscissa in these curves is the ratio of the flow velocity ahead of the shock to the small amplitude disturbance speed and not the ordinary sound speed. This difference in the definition of the Mach number accounts for the fact that the curves are not identical in the range in which they satisfy the ordinary shock equations. In the range of the switch-on shock waves both the temperature and density behind the shock are lower than they would be for the corresponding ordinary shock. As in the previous case this is due to the magnetic energy density and magnetic pressure behind the shock.

The magnitude of the tangential component of magnetic field which is switched on in the shock wave must be zero both for weak shock waves $M \to 1$ and for the critical Mach number at which the transition from the ordinary shock to the switch-on solution occurs. Thus, the magnitude of the tangential component of magnetic field behind the shock has a maximum at some intermediate shock Mach number. It may be seen that the magnitude of the tangential component of magnetic field can in some cases be slightly larger than the normal component of magnetic field.

In order to give some feeling for the variation with initial angle of the magnetic field, the properties behind fast shock waves have been plotted in Fig. 8 with the initial angle as a parameter for the special case, $a_1/b_1 = 0$. As can be seen there are no dramatic new effects occurring. The various curves between 0 and 90^o fall smoothly between these limiting cases which we have already discussed. It is of some interest to note that the curves corresponding to large angles tend to bunch together. Thus, angles over a fairly wide range in the neighborhood of 90^o can be approximated fairly well by the simpler 90^o calculations. The ratio of the magnitude of the magnetic field behind the shock to that ahead is also shown. It can be seen by inspection that this is always less than the density ratio. However, as is obvious for switch-on shock waves, the ratio of the tangential component may be larger than the density ratio. In the limit of very strong shock waves the tangential component increases by the same ratio as the density. However, as the shock strength decreases the density ratio decreases while for small angles the ratio of the magnitude of magnetic field increases. Thus the ratio of the tangential component must become larger than the density ratio.

Slow Shocks

Some of the properties of slow shock waves are shown in Figs. 9 and 10. In Fig. 9 the properties of switch-off shock waves are shown as a function of the initial angle of the magnetic field relative to the wave normal for various values of a_1/b_1. Since switch-off shock waves propagate at the maximum allowable speed for slow shocks, namely the intermediate speed, the switch-off shock may be regarded as the strongest possible slow shock for a particular plasma condition. Since the limit of weak shocks was covered in the consideration of small amplitude disturbances, it would seem instructive to examine the opposite limit of strong shocks. For propagation along the magnetic field, there is no tangential component of magnetic field to switch-off. Thus, for a_1/b_1 less than unity, the shock reduces to an ordinary hydrodynamic shock with a Mach number equal to b_1/a_1. For a_1/b_1 greater than unity, the intermediate and slow speeds are equal for propagation along the magnetic field. Thus, the propagation speed of the switch-off shock must be equal to the slow small amplitude disturbance speed. The shock wave is therefore weak and the temperature and density ratios must be equal to unity. For large angles between the wave normal and the magnetic field, the propagation velocity becomes small since the intermediate speed depends on the normal component of magnetic field. In this limit, the pressure balance across the shock wave approaches a hydrostatic balance. That is, the dynamic pressure becomes small and thus, the gas pressure change must balance the change in magnetic pressure associated with the change in tangential component of magnetic field. As a_1/b_1 becomes large, the gas pressure becomes large compared to the magnetic pressure. Thus, only small density changes are required to change the gas pressure by an amount equal to the magnetic pressure, and the density ratio across the shock waves becomes small. This is exhibited in Fig. 9 by the fact that for a_1/b_1 greater than about one the density ratio across the switch-off shock at all initial angles of the magnetic field is fairly small. We may conclude that the switch-off shock and thus all slow shocks have a fairly small amplitude for a_1/b_1 greater than about one.

The change in thermal energy across the shock wave has been non-dimensionalized with respect to the available magnetic energy per particle. In the limit of propagation normal to the magnetic field for arbitrary values of a_1/b_1, and in the limit of $a_1/b_1 \rightarrow \infty$ for all angles of propagation, the change in thermal energy is precisely equal to one-half of the available magnetic energy. For other conditions, the change in thermal energy is always greater. This additional energy may be regarded as coming from the kinetic energy of the flow associated with the normal velocity ahead of the shock. In the two limiting cases just mentioned, the other half of the available magnetic energy goes into producing tangential velocity behind the shock wave.

In Fig. 10, the changes in flow properties across slow shocks for a range of shock velocities but all at an arbitrarily chosen angle of 45^o between the direction of propagation and the initial magnetic field are plotted. All of these curves stop abruptly when the switch-off shock is reached, that is when $B_{y_2}/B_{y_1} = 0$. In the discussion of small amplitude disturbances, we mentioned that in the limit of a_1/b_1 much less than one, the magnetic field is so stiff that the field lines remain straight, and thus, the disturbance may be considered as an ordinary sound wave which is constrained to move along the field lines. The same consideration applies to moderate strength shock waves. As long as the shock velocity is small compared to the intermediate propagation speed the conditions behind the shock are the same as they would be for an ordinary shock propagating along the magnetic field lines at a velocity $v_{x1}/\cos\theta_1$. As the shock speed approaches the intermediate speed, the changes in magnetic field do become significant in changing the flow properties behind the shock. Thus, for example, the density ratio appears to reach a maximum somewhat before the limiting shock strength, namely the switch-off shock, is reached.

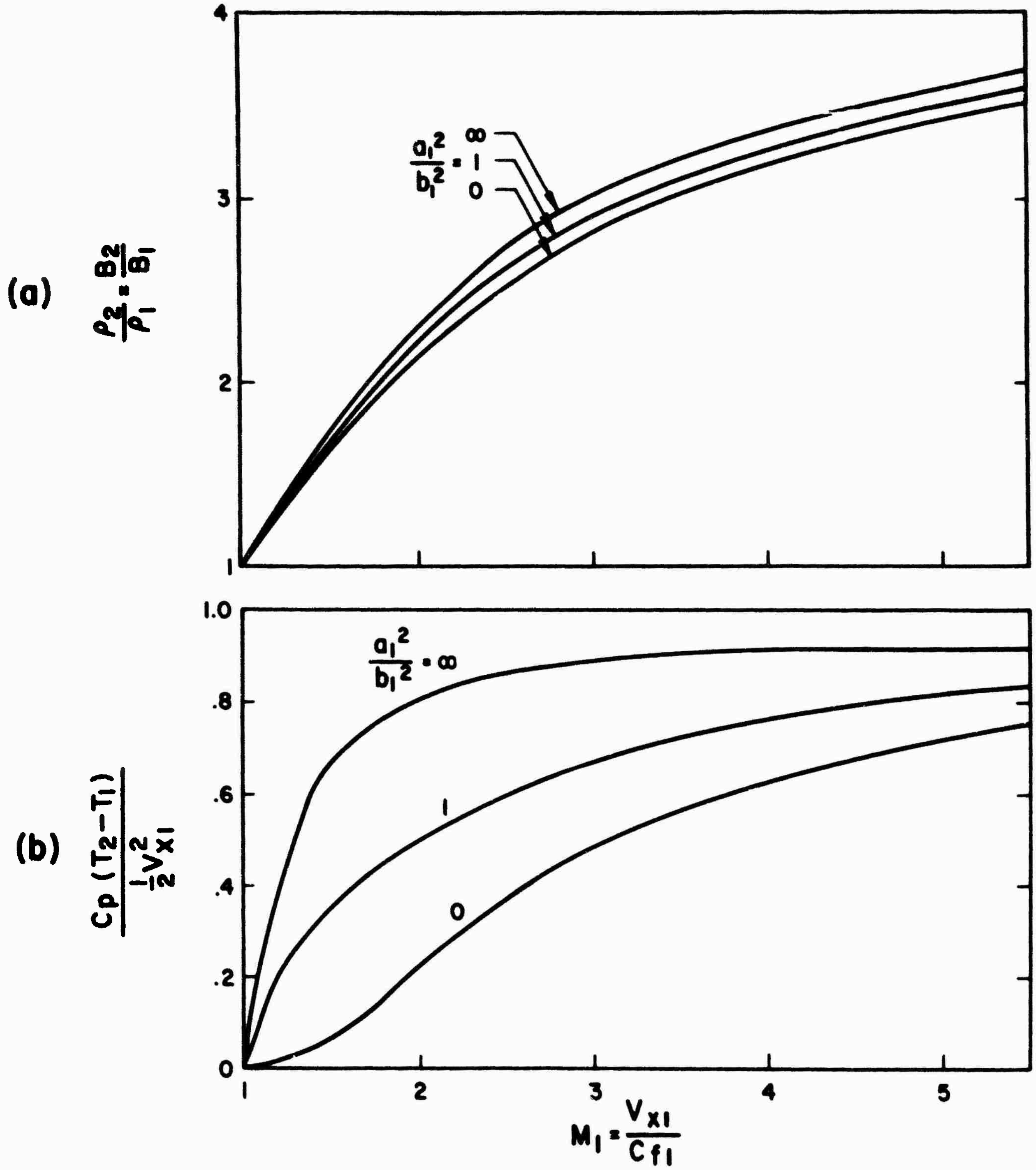

Fig. 6 Fast shocks propagating perpendicular to the magnetic field. a) density and magnetic field ratio; b) change in enthalpy normalized with respect to th flow kinetic energy ahead of the shock. Both are plotted against the shock Mach number defined as the ratio of shock velocity to the fast disturbance speed ahead. $c_{f1} = \sqrt{a_1^2 + b_1^2}$ for this case. ($\gamma = \frac{5}{3}$)

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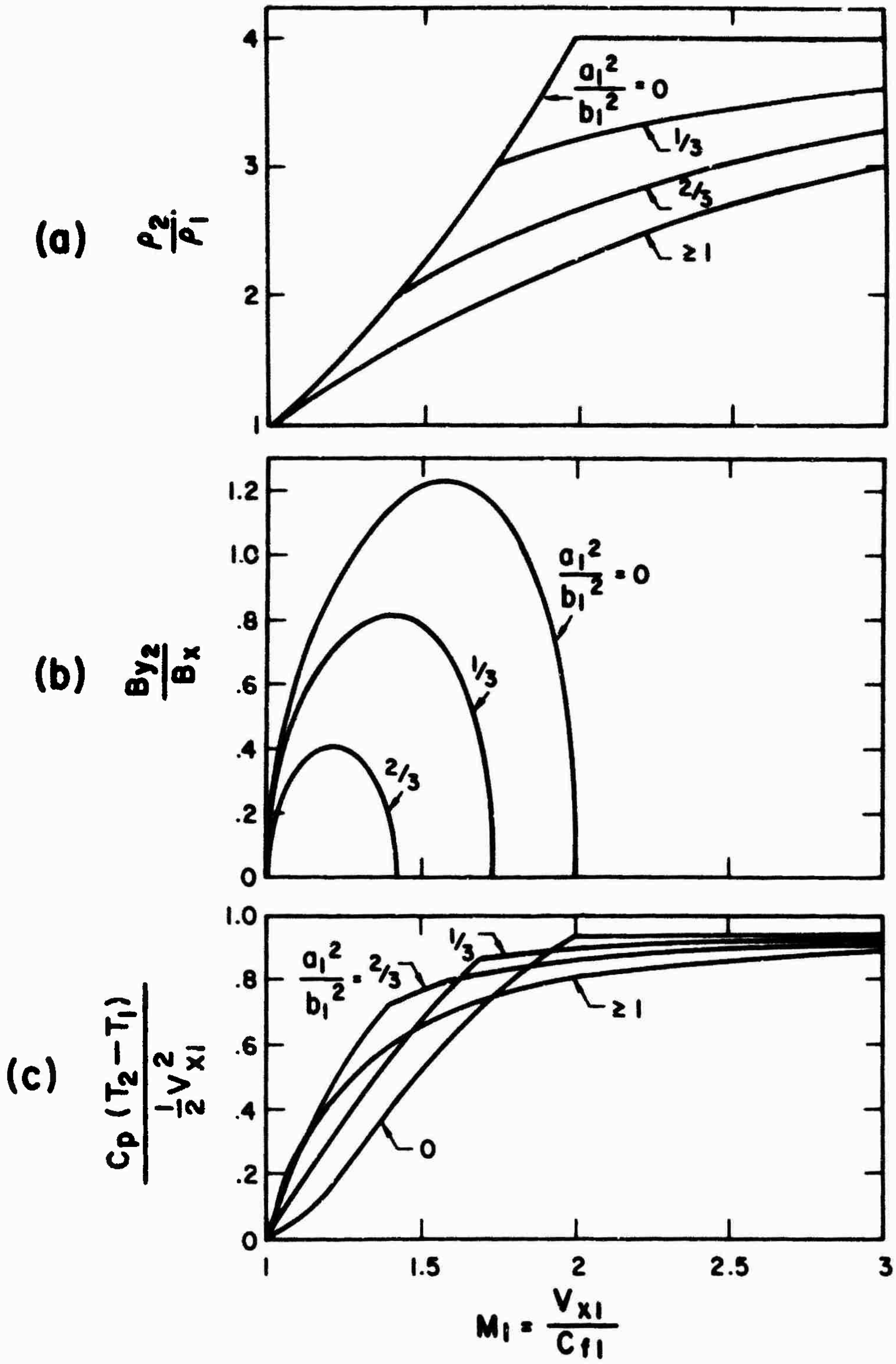

Fig. 7 Fast shocks propagating along the magnetic field. a) density ratio, b) ratio of tangential component of magnetic field behind the shock to normal component, c) change in enthalpy all plotted against shock Mach number. In this case the fast disturbance speed ahead is given by $c_{f1} = a_1$ for $a_1/b_1 > 1$ and $c_{f1} = b_1$ for $a_1/b_1 < 1$. $(\gamma = \frac{5}{3})$.

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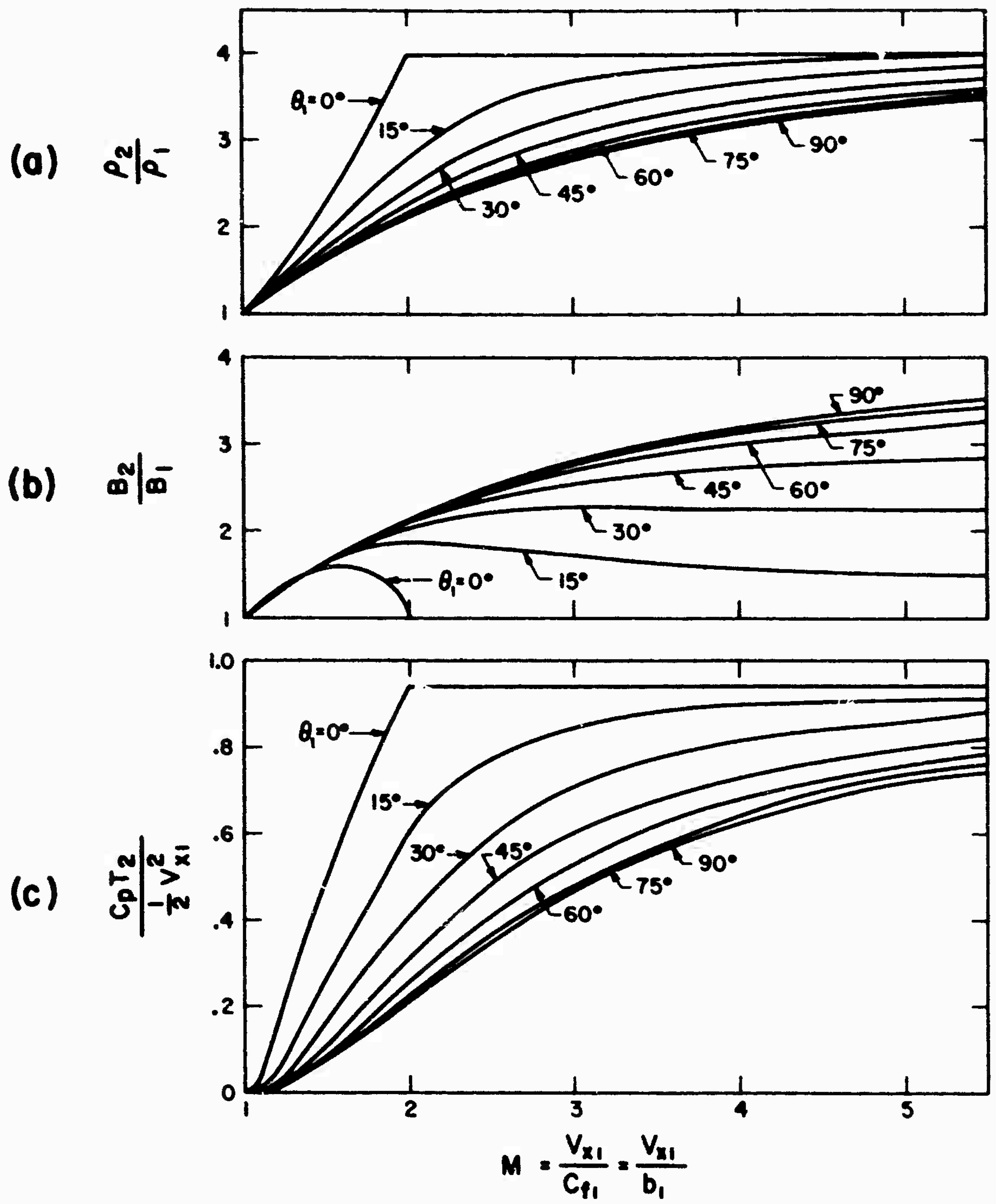

Fig. 8 Fast shocks propagating into a cold gas $a_1/b_1 = 0$ for arbitrary propagation direction relative to the magnetic field. a) density ratio, b) ratio of magnitude of magnetic field, c) enthalpy behind shock all plotted against shock Mach number. $\Theta_1 = \tan^{-1} \frac{B_{y1}}{B_x}$. $(\gamma = \frac{5}{3})$.

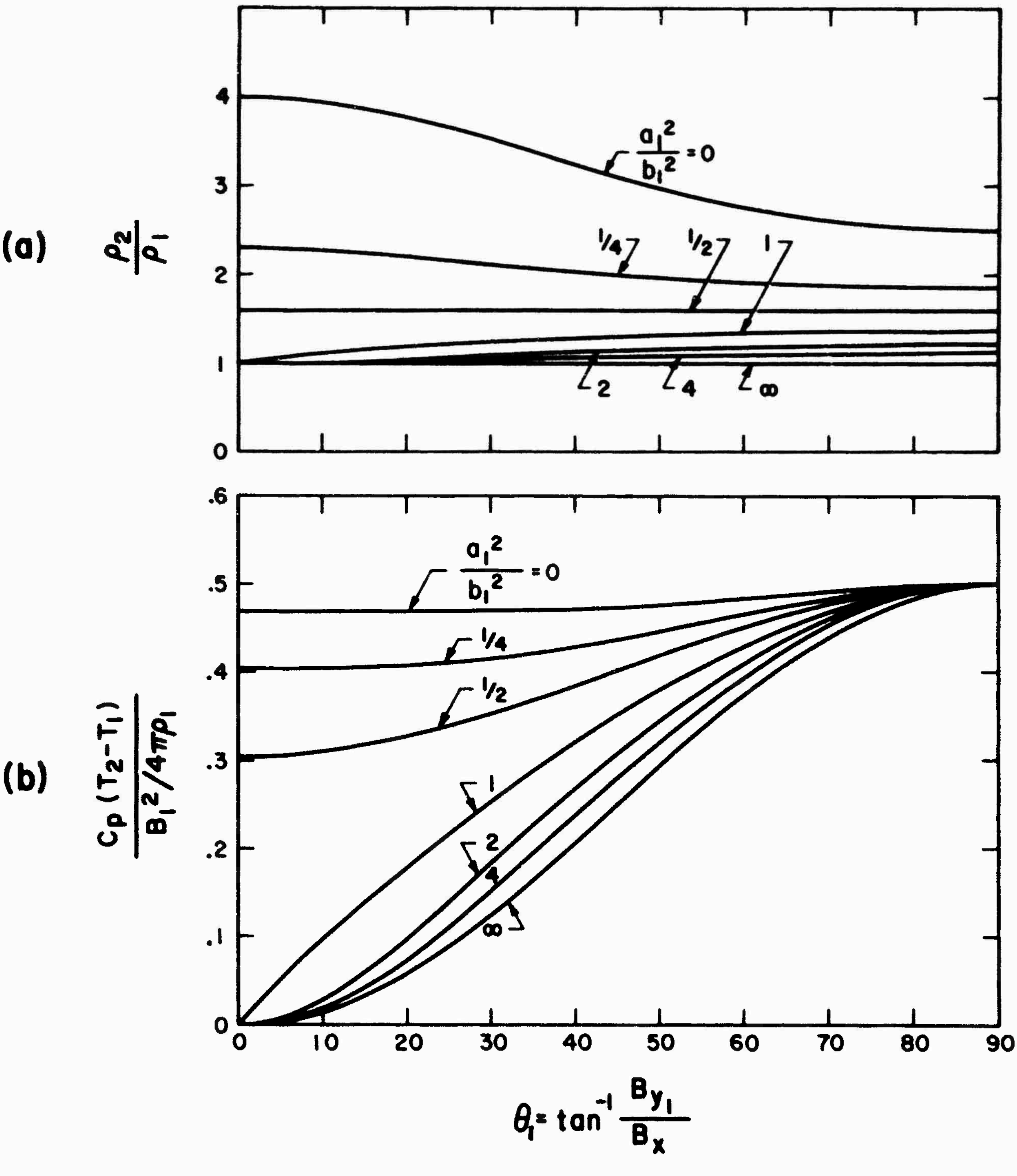

Fig. 9 Maximum strength slow shocks (switch off, $B_{y2} = 0$)
a) density ratio, b) change in enthalpy normalized with respect to available magnetic energy per unit mass plotted against direction of magnetic field relative to wave normal. ($\gamma = \frac{5}{3}$).

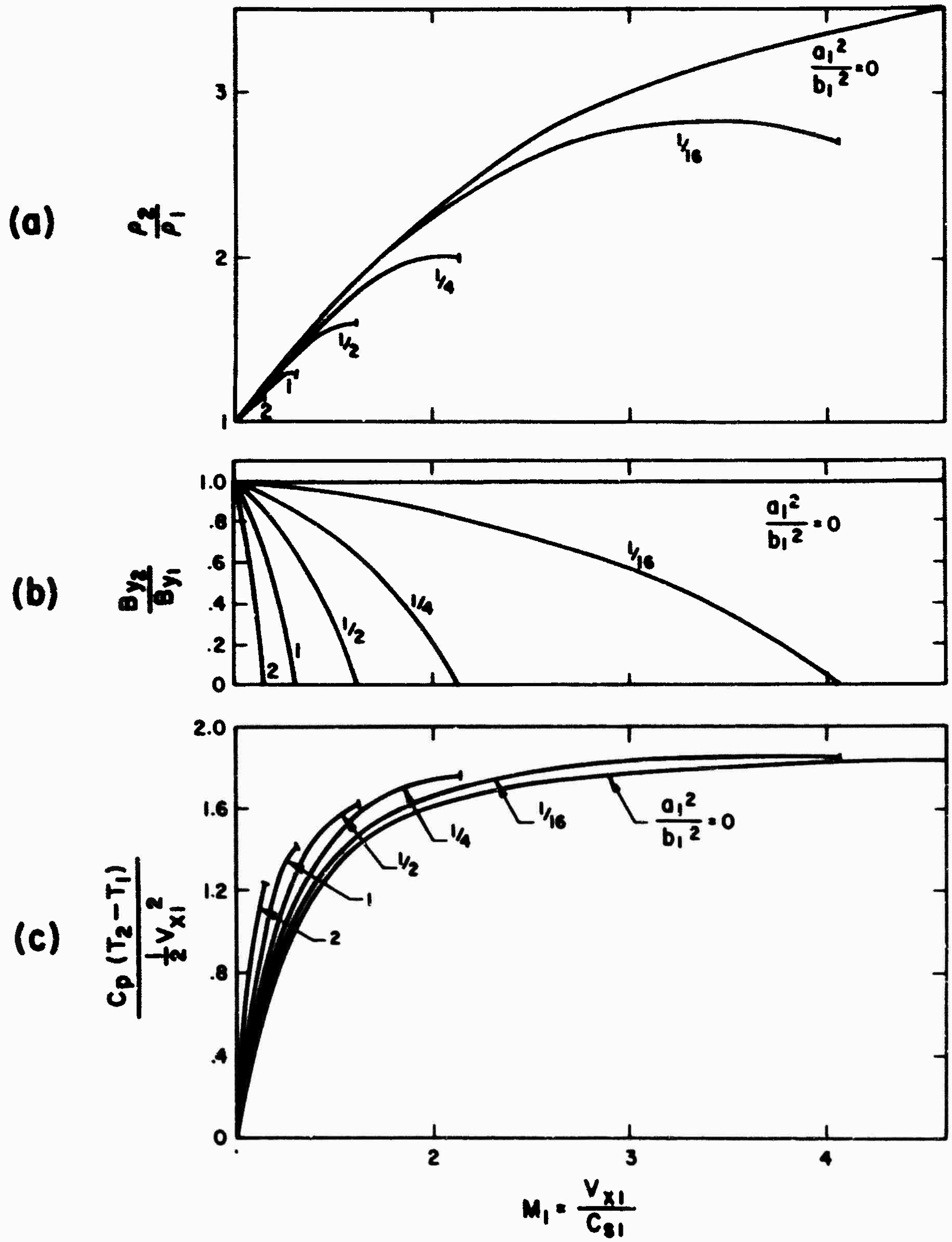

Fig. 10 Slow shocks propagating at 45° relative to the magnetic field direction. a) density ratio, b) ratio of tangential component of magnetic field, c) change in enthalpy all plotted against Mach number, ratio of shock velocity to slow disturbance speed. ($\gamma = \frac{5}{3}$).

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SECTION V

SHOCK STRUCTURE

The structure of shock waves depends not only upon the various parameters used to describe the macroscopic properties of the shock waves in the previous section, but also upon the plasma conditions, such as degree of ionization and the ratio of the mean free path to the gyro-radius. The entire subject, therefore, covers a vast range of phenomena, many of which are not fully understood at the present time. A complete discussion of the present state of knowledge on this subject is beyond the scope of this chapter. We shall restrict ourselves to a very brief and highly simplified discussion of some of the salient features.

There are two predominant reasons for interest in the structure of shock waves. On the one hand all of the theory described in the earlier part of this chapter depends upon the assumption that the shock waves which are formed in the flow can be considered as extremely thin. We must therefore, have some estimate of the thickness of the shock wave in order to determine under what conditions this assumption is justified. On the other hand, there are many conditions in plasmas, particularly when the mean free path becomes large compared to the gyro-radius, where the basic dissipation mechanisms which occur in the plasma are relatively poorly understood. Since the conservation equations for shock waves require the existence of dissipation, the study of the shock structure can help to elucidate the basic dissipation processes which occur.

Collision-Dominated Shock Waves

Let us consider first cases in which the transport coefficients are assumed to be known on the basis of the usual kinetic theory arguments. This assumption can be reasonably well justified in the case of high densities and low temperatures where the mean free path is small compared to the gyro-radius. However, as we shall see later when the mean free path becomes large the usual kinetic theory approximations are no longer justified for shock waves.

The formal procedure for solving the shock structure when the transport coefficients are known involves writing down the steady flow equations including the transport terms. Given a uniform flow of known conditions at minus infinity, these equations allow a transition to only one other uniform flow condition, corresponding to the other side of the shock wave. The solution of the equations then determines the detailed shock structure. For the present discussion we shall not elaborate on this procedure. We shall restrict ourselves to rough arguments which give an estimate of the overall thickness of the shock wave.

Since the total rate of dissipation per unit area of the shock front is specified by the conservation equations, the shock structure must adjust to a thickness such that the required dissipation is produced. The rate of dissipation per unit volume is usually proportional to the square of the gradient. For example, viscous dissipation depends upon the square of the velocity gradient and joule heating depends upon the square of the current, which is equivalent to the square of the magnetic field gradient. Since the changes in velocity and magnetic field across the shock are known, the volume rate of dissipation varies inversely as the square of the shock thickness. Since this dissipation exists over a region of the order of the shock thickness the total rate of dissipation within the shock front varies inversely as the first power of the shock thickness. Thus, as a pressure pulse steepens to form a shock wave, the total dissipation rate is initially low and the steepening process continues until a thickness is reached such that a sufficient dissipation is produced. On the other hand, if we imagined a very sharp discontinuity to be formed initially which is thinner than the required shock structure, then the dissipation would be too high and the discontinuity would spread out until the steady state shock structure is reached.

Let us first obtain a very crude estimate of the shock thickness in the case in which viscosity is the predominant dissipation mechanism. If Φ is the dissipation rate per unit area of the shock front divided by the rate of flow of kinetic energy ahead of the shock wave, then the above remarks lead to the conclusion

$$\Phi \approx \frac{\mu\left(\frac{\mathbf{v}_2-\mathbf{v}_1}{\delta}\right)^2\delta}{\frac{1}{2}\,\rho_1\, v^3_{x1}} \approx \frac{\mu}{\rho v_{x1}\,\delta}\; 2\;\frac{(\mathbf{v}_2-\mathbf{v}_1)^2}{v^2_{x1}} \qquad \text{(V-1)}$$

where μ is the viscosity and δ the shock thickness. Since both Φ and the bracket on the right-hand side are known in terms of the overall shock conditions, the above relation gives the Reynolds number, $\rho v_{x1}\delta/\mu$, based on the shock thickness in terms of the overall shock properties. This equation therefore gives an estimate of the shock thickness.

For a strong fast shock, the energy dissipated is essentially all of the thermal energy in the gas behind the shock. This in turn is roughly equal to the kinetic energy of the flow ahead of the shock. Since the change in velocity is of the order of the velocity ahead of the shock, Eq. (V-1) reduces to the statement that the Reynolds number based on the shock thickness is of the order of unity. Using the kinetic theory formula for the viscosity the statement is equivalent to the statement that the shock thickness is of the order of the mean free path. For weaker shock waves the energy dissipated decreases more rapidly than the change in velocity across the shock wave. Thus, the Reynolds number, or the shock thickness measured in mean free paths, becomes larger as the shock strength decreases.

The same argument applied to the case in which joule heating is the predominant dissipation process leads to the relation

$$\Phi \approx \frac{c_\epsilon{}^2 \delta}{(4\pi)^2\sigma} \left(\frac{B_{2y} - B_{1y}}{\delta}\right)^2 \Bigg/ \frac{1}{2} \rho_1 v_{x1}^3 \approx \frac{c_e{}^2}{4\pi\sigma v_{x1}\delta} \frac{B_{y1}{}^2}{2\pi\rho v_{x1}{}^2} \left(\frac{B_{y2} - B_{y1}}{B_{y1}}\right)^2 \qquad \text{(V-2)}$$

where c_e is the velocity of light and σ the electrical conductivity. This relation specifies the order of magnitude of the magnetic Reynolds number $4\pi\sigma v_{x1}\delta/c_e{}^2$, which would be required to provide the appropriate dissipation in terms of the macroscopic shock parameters. The shock thickness can therefore be estimated from Eq. (V-2) if the predominant dissipation process is joule heating.

If both dissipation coefficients are finite, one might expect that the appropriate shock thickness is simply given by choosing the larger of the two thicknesses given by Eqs. (V-1) and (V-2). While this view is correct under some conditions, in general it is somewhat oversimplified. For example, in a strong fast shock propagating in a plasma with a low electrical conductivity ($4\pi\sigma\mu/\rho c_e{}^2 << 1$), there are actually two characteristic thicknesses associated with the shock. Near the front of the shock, the magnetic field rises with very little change in flow velocity in a distance such that the magnetic Reynolds number is of the order of unity. Following this there is a more abrupt change in the flow velocity with very little change in the magnetic field. The change in flow velocity occurs in a region whose thickness is such as to make the ordinary Reynolds number of the order of unity. A rough criterion for the conditions under which both dissipation mechanisms are important in determining the shock structure may be obtained as follows. For a fast shock propagating perpendicular to the magnetic field the current at any point within the shock is proportional to the electric field in a coordinate system moving with the gas and may be written as

$$j = \frac{\sigma}{c_e} (v_{x1} B_{y1} - v_x B_y) \qquad \text{(V-3)}$$

Since the flow velocity decreases monotonically, through the shock, the current density will always be less than what is obtained by replacing v_x by v_{x1}. Since the current density is the curl of the magnetic field we may rewrite Eq. (V-3) as

$$\frac{1}{\delta} = \frac{1}{B_y - B_{y1}} \frac{d(B_y - B_{y1})}{dx} \leq \frac{4\pi\sigma v_{x1}}{c_e{}^2} \qquad \text{(V-4)}$$

This equation then states that the minimum distance in which the magnetic field can rise is such that the magnetic Reynolds number based on that distance be of the order of unity. Returning to Eq. (V-2), we see that for strong shocks the thickness required to produce all of the dissipation in the shock wave is less than this minimum thickness. Thus, for strong shock waves, joule heating alone is incapable of producing the required amount of dissipation. When this occurs, there must also be a portion of the shock wave in which the velocity gradients are sufficiently steep so that viscous dissipation can account for the remainder of the required dissipation.

A somewhat more precise cirterion for when both dissipation mechanisms are required in the shock structure can be obtained as follows. Let us look at the shock structure from the viewpoint of the shock formation process. We know that for a fast shock, a small amplitude disturbance behind this shock propagating at the fast propagation speed will overtake the shock. However, as the gradients in the wave that is catching up increase, the magnetic field will no longer follow the density changes and the propagation speed of the wave will decrease until it becomes equal to the ordinary hydrodynamic sound speed. A wave with such a steep gradient may or may not be able to overtake the shock depending upon whether the flow velocity behind the shock is less than or greater than the ordinary sound speed. For the case in which the flow velocity behind the shock is greater than the ordinary sound speed, the steepening process would stop when gradients are reached such that the propagation speed is reduced to the flow speed. In this case one would not obtain gradients so steep that viscous dissipation is required within the shock structure. On the other hand, if the flow velocity is less than the ordinary sound speed, the steepening process can continue even when the gradients have a characteristic length so short that there is no change in magnetic field. The steepening would then continue until another dissipation mechanism, such as viscosity, inhibits further steepening. Thus, in this case there would be two characteristic lengths associated with the shock structure as discussed above. Applying this criterion to a shock wave propagating perpendicular to the magnetic field into a plasma with $a_1/b_1 = 0$ we find that the shock structure can be based purely on joule dissipation for Mach numbers less than about two while for larger Mach numbers viscous dissipation is also required. As a_1/b_1 increases this critical Mach number is reduced.

The above discussion has been extremely restricted. We have considered the competition between two dissipation processes only for the case of fast shocks. Furthermore, we have considered only two possible dissipation mechanisms; electrical conductivity and viscosity. In general, other dissipation processes such as heat conduction, temperature relaxation between electrons and ions, collisions with neutrals, and ionization of neutrals may also be important. Conditions such as Eqs. (V-1) and (V-2) with the appropriate dissipation mechanism can be useful in giving an order of magnitude estimate of the shock thickness. Caution should, however, be used to insure that the dissipation process which is being considered can indeed provide all of the required dissipation.

Collision Free Shock Waves

Let us now turn to a case in which ordinary dissipation mechanisms associated with interparticle collisions cannot produce the required dissipation in the shock wave. Consider a plasma in which $a/b \ll 1$ but the temperature is high enough so that the mean free path is very large compared to the ion-gyroradius. For a fast wave propagating through such a medium the changes in gas pressure will be small as compared to the changes in dynamic pressure and magnetic pressure. To a first approximation we may therefore neglect the plasma pressure entirely. Within this approximation the relations determining the small amplitude wave propagation are independent of whether or not the length scale is larger or smaller than the mean free path. For length scales in the flow field larger than the mean free path the distribution function of particle velocities will acutally be isotropic and Eqs. (II-1) through (II-5) will be valid in detail. For lengths scales small compared to the mean free path the pressure in Eq. (II-2) should become a tensor and Eq. (II-4) becomes invalid. However, if the pressure is sufficiently small it can be neglected in Eq. (II-2) whether it is a tensor or a scalar. Furthermore, in this case Eq. (II-4) is no longer required. Thus, the relations determining the propagation of fast waves are independent of whether the length scale is larger than or smaller than the mean free path. It follows immediately that the arguments which were used to show that a compression pulse steepens towards a shock wave are valid even when the length scale of the compression pulse is small compared to the mean free path. We may therefore expect that the shock wave which forms will have a structure containing length scales much smaller than the mean free path. As a result binary collisions will probably not provide the required dissipation in the shock.

The length scale at which the above argument for steepening ceases to apply is the ion-gyroradius based on a velocity equal to the Alfven speed. We may therefore anticipate that the actual shock structure will contain length scales of this order of magnitude. This limitation arises from the fact that Eq. (II-3) becomes invalid when such steep gradients are reached. (See A-9 and A-10). This equation is based on the assumption that $\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}/c_e = 0$. In the absence of collisions this is merely the statement that individual particles drift at the E/B velocity. This, is however, only valid as long as the gyro-radius is small compared to the length scale of the changes in the magnetic field, i.e., when the ions move adiabatically. Thus, when length scales comparable to the ion-gyroradius are reached the basic equations become invalid and a different dispersion relation is obtained. The steepening process is therefore modified at this point.

Several attempts at theoretical predictions of the structure of such shock waves have been made. A clear-cut resolution of the differences between these various theories and the range of plasma conditions over which they apply must await more detailed experimental evidence. In all of these theories the shock wave contains a fine structure with scales of the order of the ion-gyroradius or less. For weak shock waves it is possible to obtain a fairly regular solution consisting of a long train of large amplitude waves which are steady in a coordinate system moving with the shock wave. In the presence of any finite rate of damping due to either collisions or Landau damping these oscillations will eventually damp to the uniform conditions behind the shock wave. The scale length of the individual waves is of the order of the ion-gyroradius for general directions of propagation of the shock wave relative to the magnetic field. However, for the special case, which has received the most attention, of propagation perpendicular to the magnetic field the scale length is smaller than this by the square root of the mass ratio between electrons and ions.

For strong shock waves the microstructure is probably better characterized as a random turbulent structure. In this case, the flow energy goes initially into turbulent motions and later damps into actual thermal velocities of the ions and electrons. For such a turbulent shock wave the changes in the average density, average flow velocity and the average magnetic field strength can occur in a distance in which the energy which must be dissipated in the shock wave is put into some form of random energy. Thus, the thickness of the shock wave as defined by the average quantities does not have to be as long as the thickness required for the turbulent energy to damp into particle motions. It is therefore possible to have a shock wave in which the major changes in density, flow velocity, and magnetic field occur in a region in which the random energy goes into turbulent magnetic energy. The region in which this turbulent energy is damped into particle motions may be significantly larger but have only small changes in density, flow velocity, and average magnetic field associated with it.

The best experimental evidence for collision free shock waves at the present time has been obtained from recent measurements on the IMP satellite.[5] The interplanetary plasma has a flow velocity which is of the order of 5 to 10 times the fast propagation speed. The mean free path in this plasma is of the order of 10^8 kilometers. The interaction of this wind with the earth's magnetic field is found to produce a bow shock wave with a thickness of the order of 1000 kilometers or less. This thickness is, therefore, many orders of magnitude less than the mean free path and is not too different from the ion-gyroradius, which is of the order of 100 kilometers. These results are fairly recent and it is not clear that the experiment has sufficient resolution to have observed a shock wave much thinner. The shock wave is characterized by a sudden jump in the average magnetic field strength as well as an increase in the turbulent fluctuations in the magnetic field strength.

SECTION VI

APPLICATIONS

In this section we will cite briefly two examples which illustrate applications of the theory discussed in previous sections. We shall also discuss briefly some experimental evidence which supports the theory. Unfortunately, at the present time there is only very little experimental data available.

Magnetic Shock Tubes

The distance time diagram shown in Fig. 5(b) corresponds to a fast shock followed by a uniform region and a slow expansion fan. If such a flow configuration can be produced in the laboratory, the uniform gas sample between the fast and slow waves provides a possibility of achieving a high temperature plasma of known conditions for study in the laboratory. Plasma conditions behind the shock are known in terms of the shock velocity and the conditions ahead of the shock. Thus, the relatively simple measurement of a shock velocity determines the average magnetic field, enthalpy and density in the hot plasma sample.

A schematic diagram of the magnetic annular shock tube in which such a flow has been achieved is shown in Fig. 11. Initially a gas and a quasi-steady magnetic field are present in the thin annular region between the conducting cylinders. When the condenser bank is discharged the magnetic field at the insulator is changed and waves propagate along the device. If the condenser bank is arranged such that the current is essentially a step function then the diagram shown in Fig. 5(b) (with no intermediate wave) with plasma conditions a function of x/t only is appropriate. The boundary conditions which must be applied are that for an ideal insulator there is no mass flow through the insulating surface and that the tangential component of magnetic field at the insulating surface is specified by the known current from the condenser bank. The current from the condenser bank flows partially in the shock wave and partially in the slow expansion fan.

It has been demonstrated that under appropriate conditions, a shock wave is produced which travels at the calculated velocity, that there is a region of uniform plasma flow behind the shock wave, and that the density and magnetic field strength in this region corresponds to the calculated conditions behind the shock.[6] Some uncertainties still exist as to the temperature behind the shock and the expansion fan has not been clearly identified.

It should be pointed out that a number of criteria must be satisfied in order to achieve such operation. Most experiments which attempt to produce magnetically driven shock waves do not produce a clean shock with a uniform test sample behind it. First the shock velocity must be sufficiently high so that the gas behind does in fact, have a high electrical conductivity. For low conductivity behind the shock the driving magnetic field will diffuse through the uniform test sample and thus disturb both the uniform region and the shock characteristics. In the experiments mentioned above the gas ahead of the shock wave was at room temperature. It was found that for sufficiently strong shock waves the electric field ahead of the shock was small enough to justify the assumption that $\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}/c_e$ was zero ahead of the shock. It has been found by several authors[7] that when one attempts to produce a slower shock the electric field ahead of the shock is no longer negligible compared to the cross product of the flow velocity and magnetic field, ahead of the shock. In this case, other modes of operation are found. It is for example, no longer necessary that the initial disturbance travel at a speed greater than the fast propagation speed.

A second criterion for reasonable operation of such a device, is that ablation does not occur.[8] Ablation which seems to be predominantly associated with the insulating wall adds mass to the region between the cylinders and thus can significantly slow down the resulting shock wave.

A third criterion is that the annular region between the cylinders be sufficiently small compared to the radius of the cylinders.[9] The azimuthal component of magnetic field which is to be regarded as the y-component in our previous discussions will for a force-free field fall off inversely as the radius. Thus, there is a difference between this field at the inner and the outer walls. Unless the radius ratio is kept sufficiently close to unity this nonuniformity will destroy the one-dimensional nature of the flow.

A series of experiments has also been conducted which have verified the linear propagation of the fast mode propagating along the magnetic field in a plasma with a/b small compared to one.[10] These experiments have recently been extended to a larger amplitude disturbance and have shown that such large amplitude pulses do indeed steepen towards shock waves.[11]

Conversion of Magnetic to Plasma Energy

It was mentioned in the discussion of slow shock waves that these shock waves tend to convert magnetic energy to plasma kinetic or thermal energy. This may be a significant mechanism by which energy is converted in a number of situations which may occur in nature. Let us imagine an interface between two regions of plasma in which the magnetic field has significantly different orientation. The overall pressure balance across the interface will be adjusted rapidly by the fast wave mode. If there is no component of magnetic field normal to the surface then the intermediate and slow propagation speeds are zero and one would expect the magnetic fields on the two sides to diffuse into one another at a speed determined by the conductivity of the medium. If, on the other hand, a small component of magnetic field normal to the interface exists, then the propagation of intermediate and slow waves is possible. It is always possible to find a combination of intermediate and slow waves of appropriate strengths propagating in both directions from the boundary to satisfy the boundary conditions of appropriate change in direction and magnitude of the magnetic field. Furthermore, since the components of magnetic field which are oppositely directed on the two sides of the boundary can cancel one another, the resulting configuration will have a lower magnetic energy. For a high conductivity medium the wave propagation process will obviously convert the magnetic energy at a much higher rate than the diffusion process since the wave propagation speeds do not decrease with increasing conductivity. In the example we are presently discussing the waves would propagate at a constant speed and the region in which the magnetic energy has decreased would increase linearly with time.

It has also been suggested[12] that a steady two-dimensional configuration is possible in which the predominant mechanism for the conversion of magnetic energy to plasma energy is the existance of standing intermediate and slow waves. In this case an x-type neutral point is formed in the flow. The flow on both sides of the boundary is towards the boundary and leaves the region by flowing along the boundary, in opposite directions on either side of the neutral point. In such a configuration standing waves are possible along the boundary with the exception of a small region in the immediate vicinity of the neutral point. In this region since the magnetic field is small the propagation speed vanishes and therefore diffusion must still be important. However, the region over which diffusion is important is significantly smaller than it would be if one had assumed no normal component of magnetic field anywhere along the boundary. As a result, the net rate at which flow goes towards the boundary and the rate at which magnetic energy is converted to plasma energy decrease only logarithmically with increasing conductivity or magnetic Reynolds number. Since pure diffusion would lead to an inverse square root dependence on conductivity, the wave mechanism leads to significantly higher rates for high magnetic Reynolds number situations.

This result is probably of particular significance in a number of astrophysical situations where the length scale is so large that the magnetic Reynolds number is usually very large. Several examples where such a flow configuration may be of interest can be given. The origin of solar flares is believed to be a storage of magnetic energy above the photosphere of the sun which is then released suddenly. A rapid mechanism for conversion of this magnetic energy such as the one just described seems to be required in order to account for the observed suddenness of solar flares. At the boundary of the magnetosphere the magnetic field direction on the solar wind side and on the earth's side are significantly different. As a result one may expect that the boundary would resolve itself into a combination of intermediate and slow waves. This in turn would result in a significant rate of reconnection of the field lines in the solar wind to the dipole field lines on the earth, which, in turn can cause motions within the magnetosphere. Thus far, there have been no direct measurements from satellites at boundary crossings to indicate this resolution of the boundary. However, the flow rate observed indirectly inside the magnetosphere from ionspheric currents and auroral motions is in rough agreement with the predicted rate of reconnection at the boundary. Finally, a dilemma is raised by the fact that the solar wind continually drags field lines away from the sun, however, the net field strength at the surface of the sun cannot increase indefinitely. Thus, these field lines must eventually be broken so that they may return to the sun. On the basis of simple diffusion arguments this rate of breaking would be much too slow. It seems likely that a mechanism involving wave propagation such as the one just discussed could give a sufficiently rapid rate of breaking to avoid this increase in field strength at the solar surface.

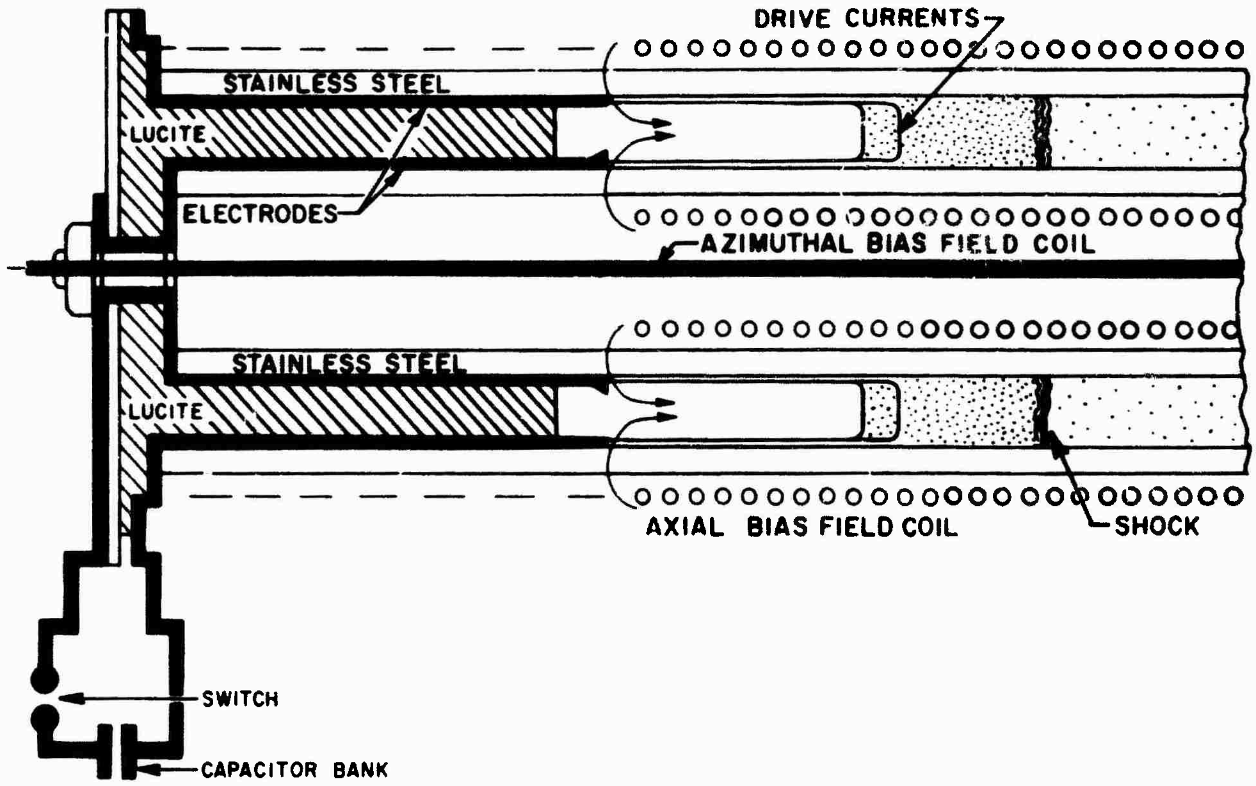

Fig. 11 Schematic cross-sectional view of magnetic annular shock tube.

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APPENDIX

RANGE OF APPLICABILITY OF BASIC EQUATIONS

The limitations on the validity of the basic Eqs. (II-1) to (II-5) arise principally from two causes. First, since heat conduction and viscosity were neglected, the process by which the particle distribution function is made isotropic must be rapid compared to the typical time scale in the flow. Secondly, the equations assume infinite conductivity thus, implying that currents can flow freely. If the predominant process for the randomization of the particle distributions is scattering by binary collisions these two requirements at first sight seem somewhat contradictory. The requirement of rapid achievement of isotropy implies a mean free time for collisions short compared to the flow time, while the requirement of high conductivity implies a long mean free time. As we shall see, these two requirements are, however, not mutually exclusive. Thus, for a fully ionized plasma of a given density and with a given length scale to the flow field, there exists a minimum temperature below which the conductivity becomes too low and a maximum temperature above which the mean free time becomes too large for isotropization to be achieved sufficiently rapidly by binary collisions.

These limits will be evaluated quantitatively below. The actual range of validity of the equations is, however, probably considerably larger than implied by these limits. Several types of nonisotropic particle distributions in collision-free plasmas are known to be unstable. As such an instability grows, it must lead to the production of a more isotropic particle distribution. Thus, if the growth times of these instabilities is sufficiently short, isotropy of the particle distribution may be achieved by this mechanism rather than by binary collisions. This would imply that the above equations may be valid at temperatures or mean free times longer than the limits derived from assuming randomization by binary collisions.

At the present time, no concise and quantitative estimate of the limits to the validity of the equations based on randomization by the growth of instabilities has been given. It should, however, be borne in mind that the limits which we will now derive based on particle collisions probably underestimate the region of validity.

Let us now turn to discussion of the limitations on the individual equations. Equations (II-1) and (II-5) are generally valid. In the momentum equation, Eq. (II-2), the relation

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \frac{4\pi \underset{\sim}{j}}{c_e} \qquad \text{(A-1)}$$

has been used to eliminate the current density $\underset{\sim}{j}$. As before, c_e is the velocity of light. Thus, the assumption has been made that the displacement current is negligible. This implies that both the fluid velocities and the wave velocities must be small compared to the velocity of light. Furthermore, the body force due to the interaction of the electric field with any net charge density in the plasma has been neglected. This assumption is valid if the Debye length is small compared to the length scale of the flow field. Neither of the above assumptions are very restrictive for the range of plasma conditions which are usually of interest. In writing the pressure as a scalar we have assumed that the gradients are sufficiently gentle so that viscous terms are not important. Quantitatively, this condition may be given as

$$\mu \frac{v}{L} \ll p \quad \text{or} \quad \lambda \ll L \tag{A-2}$$

where μ is the viscosity of the plasma, L is a characteristic length of the flow field, and λ is the mean free path for particle collisions. The second form of this restriction can be obtained from the first by making use of the ordinary kinetic theory formula for the viscosity in the absence of a magnetic field and making the assumption that the flow velocity is of the order of the thermal velocity of the ions. It could also be obtained more directly from the condition that in order to maintain the particle distribution isotropic, there must be frequent randomizing collisions and therefore the collision distance must be small compared with the distance in which the flow properties change appreciably.

We may note in passing that there are the two limits in which the above condition is overly restrictive. One of these occurs when the iongyro-radius is small compared to the mean free path, and the gradients are primarily in a direction perpendicular to the magnetic field. Under these conditions, the particle orbits will be turned in a distance appreciably less than the mean free path. The distribution function for motions perpendicular to the magnetic field then becomes isotropic in a smaller distance than the mean free path. Alternatively, we could say that under these conditions, the viscosity becomes a tensor whose components perpendicular to the magnetic field become appreciably reduced. The second limit occurs when both the dynamic pressure ρv^2, and the magnetic pressure $B^2/8\pi$, are large compared to the plasma pressure. In this case, the entire pressure tensor can be neglected in the momentum equation, and the restriction (A-2) is irrelevant. For the sake of simplicity we shall assume throughout the remainder of the discussion in this section that the three pressures above are comparable.

$$\rho v^2 \approx p \approx \frac{B^2}{8\pi} \tag{A-3}$$

and that the tangent of the angle between the magnetic field and the direction of the gradients is of order unity.

Equation (II-3) is obtained from Maxwell's equation

$$\nabla \times E = -\frac{1}{c_e}\frac{\partial B}{\partial t} \tag{A-4}$$

where the electric field E, has been eliminated by making use of the Ohm's law

$$E + \frac{v \times B}{c_e} = \sigma^{-1} \cdot j \approx 0 \tag{A-5}$$

under the assumption that the conductivity tensor σ, is so large that the right hand side may be taken to be negligibly small. If the electron gyro-frequency is small compared to the electron mean free time then the electrical conductivity is a scalar. In this case the above assumption is justified if

$$\frac{j}{\sigma} \ll \frac{v \times B}{c_e} \tag{A-6}$$

Using Eq. (A-1) to eliminate the current density this condition may be written roughly as

$$\frac{4\pi\sigma v L}{c_e^2} \gg 1 \tag{A-7}$$

On the other hand, if the electrons make many gyro orbits between collisions, then their motions are equivalent to motions of a free-electron in an electric and magnetic field. Thus, Ohm's law becomes equivalent to the statement

$$E + \frac{v_e \times B}{c_e} = 0 \tag{A-8}$$

where v_e is the electron drift velocity. This equation is equivalent to Eq. (A-5) with the right hand side set equal to zero, if the difference between the electron velocity and the fluid velocity is small compared to the fluid velocity. Making use of the fact that the current density is the product of the particle density, n, the electronic charge e, and the difference between the electron and ion velocities, and that for a fully ionized gas the ion velocity is approximately the flow velocity we can write the above condition in the following form

$$\frac{v_e - v_i}{v} \approx \frac{J}{Nev} \approx \frac{c_e B}{4\pi NevL} \approx \frac{M_i c_e v}{eBL} \quad \frac{B^2}{4\pi\rho v^2} \ll 1 \tag{A-9}$$

where M_i is the ion mass. Using Eq. (A-3) this reduces to

$$\frac{M_i c_e v}{eBL} \ll 1 \tag{A-10}$$

which states that the ion-gyroradius must be small compared to the scale length of the flow field.

Equation (II-4) is the statement that a fluid element changes its state isentropically. This implies that dissipation must be negligible. If we examine the dissipation associated with viscous stresses and joule heating, we find that it is small if the conditions (A-2) and (A-7) are satisfied. There is, however, another and somewhat more restrictive condition associated with thermal conductivity. The divergence of the heat flow vector corresponds to a heat addition to the fluid and therefore to an entropy change. We must therefore require that the divergence of the heat flux multiplied by the characteristic flow time, L/v be small compared to the thermal energy of the plasma.

$$k \frac{T}{L^2} \frac{L}{v} \ll \underset{\sim}{\rho} C_p T \tag{A-11}$$

where k is the heat conduction coefficient, C_p is the specific heat at constant pressure and T is the temperature. Making use of the kinetic theory formula for heat conduction in the absence of a magnetic field but remembering that the heat conduction is primarily due to the electrons because of their high thermal velocity, this equation can be written in the form

$$L \gg \sqrt{\frac{M_i}{M_e}} \lambda \tag{A-12}$$

This equation differs from Eq. (A-2) only by a numerical factor, however, Eq. (A-12) is the more restrictive one.

The limitations on the range of validity of the basic equations is thus given by Eqs. (A-7), (A-10) and (A-12). Making use of Eq. (A-12) and the kinetic theory formula for the electrical conductivity of the plasma, Eq. (A-7) may be written as

$$L \gg \frac{r_i^2}{\sqrt{\frac{M_i}{M_e}} \lambda} \tag{A-13}$$

where r_i is the ion gyro-radius. It is easily seen that if conditions (A-12) and (A-13) are satisfied Eq. (A-10) is automatically justified. The two remaining conditions are therefore Eqs. (A-12) and (A-13). Making use of the relation for the mean free path in a fully ionized plasma,[13] these equations may be rewritten as

$$L \gg \frac{3 \times 10^{14} T^2}{N} \tag{A-14}$$

$$L \gg \frac{2}{T^2} \tag{A-15}$$

where the length, L, is to be measured in centimeters, the temperature is in electron volts and the particle density in particles per cubic centimeter. Equation (A-15) which results from the electrical conductivity is satisfied if we are dealing with length scales of several centimeters and temperatures greater than a few electron volts. In using this condition for very large lengths, it should of course, be borne in mind that we have discussed only fully-ionized plasmas and that complete ionization is not expected at temperatures below about one electron volt. The condition (A-14) on the other hand, is violated at very high temperatures or low densities. Thus, for a particular length scale and a sufficiently high density a temperature range exists in which the equations are valid between temperature determined by Eq. (A-15) and that determined by Eq. (A-14).

As we have tried to indicate throughout this section the limitations given by Eqs. (A-14) and (A-15) should be regarded as defining the minimum range of validity of the basic equations. Several limiting cases corresponding to violation of the condition (A-3) and the assumption that the gradients are in a direction which makes an angle of the order of one radian with the magnetic field exist in which the range of validity would be wider. Furthermore the possibility that isotropic particle distributions are achieved by the growth of instabilities rather than by binary collisions suggests that the restriction given by Eq. (II-9) may in many cases by removed entirely.

In particular recent satellite data gives clear evidence that the flow of the solar wind over the magnetic field of the earth exhibits several magnetohydrodynamic phenomena in spite of the fact that the mean free path is several orders of magnitude larger than the scale of the region in which the interaction takes place.[5] The data indicates clearly that a shock wave is formed some distance ahead of the actual boundary between the solar wind plasma and the earth's magnetic field. The distance between the shock wave and the interface agrees with calculations based on a magnetohydrodynamic model. This distance also expands as one would expect as one moves to points away from the stagnation point. Within such a flow there are certainly phenomena occurring which cannot be explained directly by the magnetohydrodynamic model; for example, the existence of considerable turbulence which is presumably related directly to the dissipation in the shock and also the production of a non-Maxwellian tail of high energy particles. The magnetohydrodynamic theory does, however, seem capable of describing the gross properties of the flow field such as changes in average magnetic field, density and flow velocity.

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A review of the theory of MHD characteristics and shock waves is presented. Primary emphasis is placed on a physical discussion of the three characteristic modes and the jump conditions for the two types of shock waves which can exist. Brief discussions of shock structure, applications of the theory, and the range of applicability of the continuum equations are also given.

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ERRATA TO RESEARCH REPORT 185

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The numerical sequence of pages 16-21 was incorrect. Attached are the corrected pages.

It is also instructive to look at some of the wave properties in the limits of large and small ratios of a to b. In the limit of a >> b, the fast propagation speed, as well as the changes across the fast wave, reduce to those for an ordinary sound wave. This is to be expected since in this case the magnetic pressures are too small to play any role. The fast wave, therefore, approaches a purely longitudinal wave. In the same limit, the slow wave becomes a purely transverse wave. This can be seen most easily from Eq. (II-24) which shows that the change in the y-component of velocity becomes very large as compared with the change in the x-component of velocity. It follows physically from the fact that a very small change in the longitudinal velocity produces a change in density and, therefore, a change in gas pressure which is very large compared to the magnetic pressure. Thus, only very small changes in the longitudinal velocity are required to balance the changes in the magnetic pressure. For this wave and in this limit, the fluid may therefore be considered as virtually incompressible. It then follows from Eq. (II-23) and, as we have concluded earlier, that the slow propagation speed approaches the intermediate speed.

In the opposite limit, i.e., a << b, the waves do not break up into purely longitudinal and purely transverse. In this limit, the slow wave is most easily understood. Since, in this limit, both the gas pressure and the dynamic pressure ρv^2, are small compared to the magnetic pressure, we cannot have appreciable changes in the magnetic field across the wave. Thus, the magnetic field lines will have virtually no change in direction across the wave. The plasma flow, on the other hand, is strongly coupled to these field lines. Thus, the plasma is constrained to flow in a direction parallel to the magnetic field lines. We have already observed that the propagation speed of the slow wave in this limit is equal to the sound speed multiplied by the cosine of the angle between the magnetic field and the wave propagation direction. This then corresponds to a sound wave traveling along the magnetic field lines, and therefore moving more slowly in the direction of the wave normal. The slow wave therefore becomes purely transverse for propagation perpendicualr to the magnetic field and purely longitudinal for propagation along the magnetic field. On the other hand, since the velocity change across the fast wave is perpendicular to that across the slow wave, we conclude that in this limit the fast wave is purely longitudinal for propagation perpendicular to the magnetic field, while it is purely transverse for propagation along the magnetic field.

Summary

We may summarize the major conclusions which have been reached concerning these waves as follows:

1) There are three distinct wave propagation modes which can be conveniently classified according to the magnitude of their propagation speed as fast, intermediate and slow. The velocity changes across the three waves are mutually perpendicular.

2) For fast and slow waves, both the velocity and the magnetic field remain in the plane defined by the magnetic field ahead of the wave and the wave normal. On the other hand, for the intermediate wave both the velocity and magnetic field changes are purely in the direction perpendicular to this plane.

3) For the fast mode, the magnetic pressure increases when the density increases. For the slow mode, an increase in magnetic pressure corresponds to a decrease in density. Across an intermediate wave, neither the magnetic pressure nor the density change.

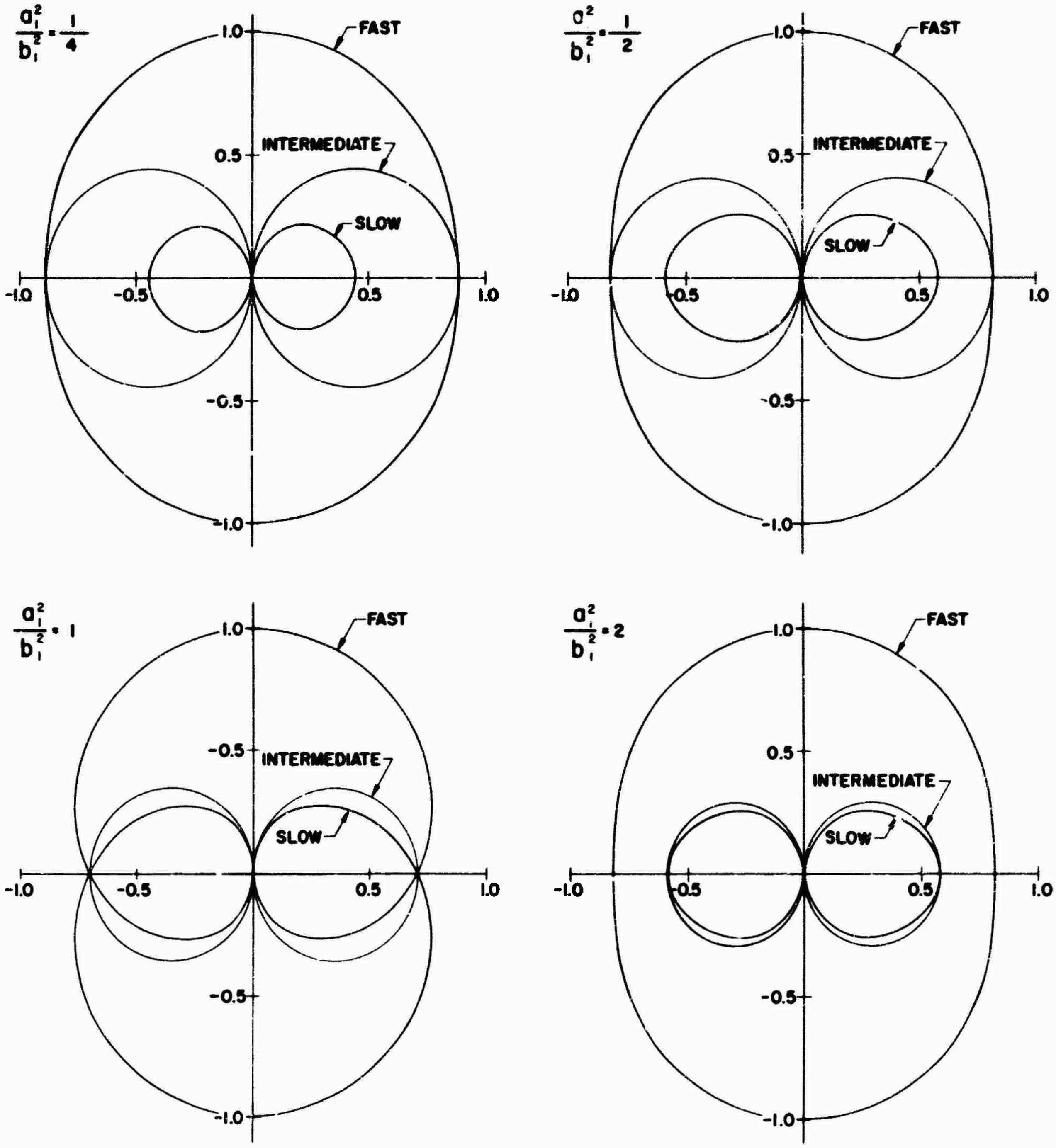

Fig. 2 Friedrichs Diagram. Polar plot showing the dependence of the propagation speeds of the three linear wave modes on the angle between the wave normal and the magnetic field. For several values of the ratio of sound speed a to Alfven speed b. Speeds have been normalized with respect to $\sqrt{a^2 + b^2}$.

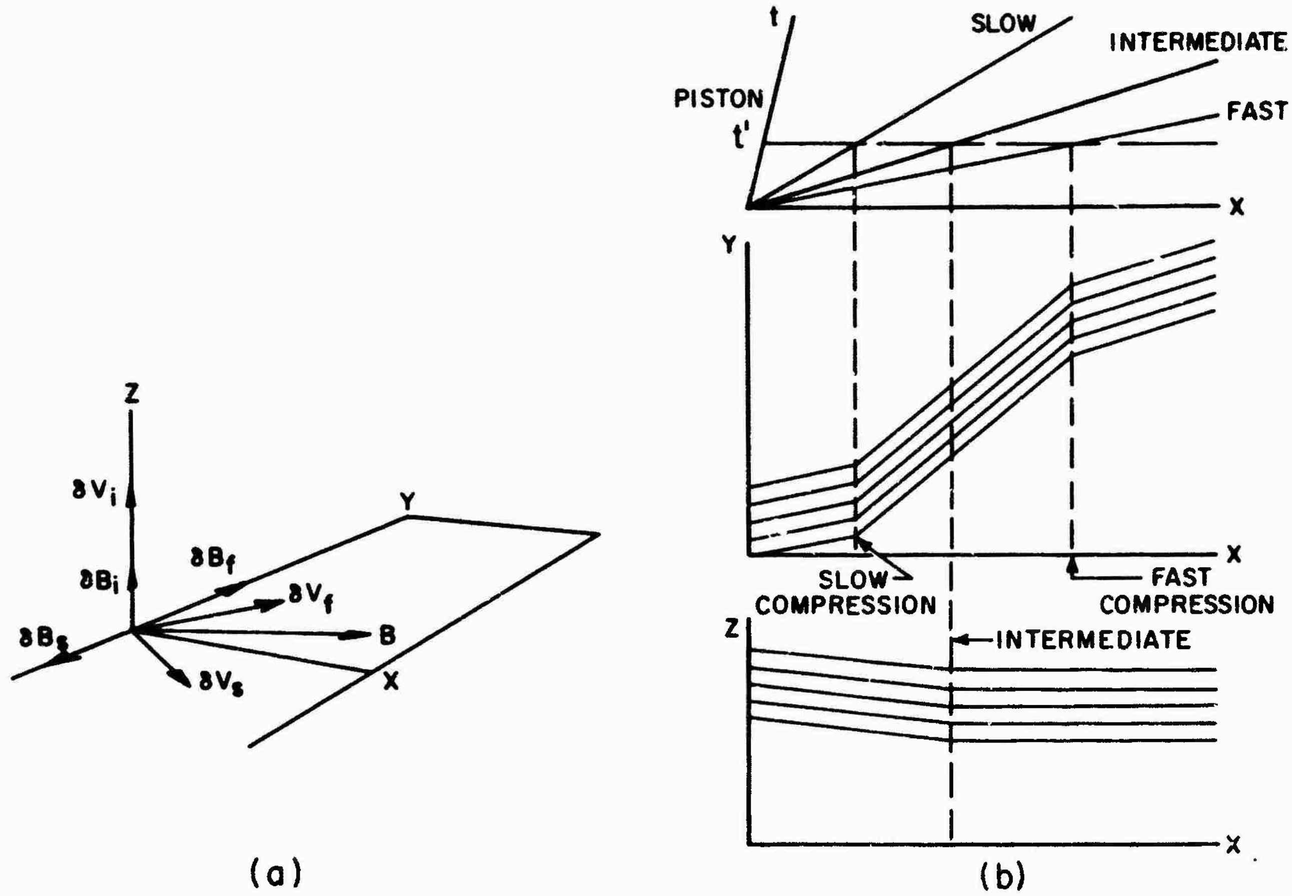

Fig. 3 Sketch of flow resulting from the instantaneous acceleration of a piston to a small velocity. In general, three waves will be emitted which separate with time as shown on the x-t diagram. The projections of the magnetic field lines on the x-y and x-z planes at a time t' are also shown for the case in which both the fast and slow waves are compressions. The changes in velocity and magnetic field across the three waves are illustrated in the vector diagrams. The initial magnetic field, δv_s and δv_f are in the x-y plane. δv_f must lie within the acute angle between the magnetic field and the y-axis. The three velocity changes are mutually perpendicular. The signs of δB_s and δB_f were also chosen for compression waves.

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SECTION III

LARGE AMPLITUDE ISENTROPIC WAVES AND SHOCK FORMATION

The solution to a nonlinear flow problem can be built up by considering it as a series of small amplitude waves, each propagating through a medium which has been modified by previous waves. In this manner, it is possible to discuss problems with arbitrarily large amplitudes. The concept of a large number of isentropic small amplitude waves describing the flow breaks down in the case where shock waves are formed. However, the nonlinear isentropic solutions can be used to predict when shock waves occur. The shock waves themselves will be discussed in the next section. In this section, we will consider the nonlinear waves related to each of the linear wave propagation modes. We will consider only the case in which the waves are all propagating in one direction, i.e., as though they were generated at the boundary of a semi-infinite plasma. For the special case in which the boundary condition is changed suddenly, the fact that the three propagation speeds are different separates the resulting nonlinear waves. Thus, for this case the nonlinear description of the individual modes can be used to obtain a general solution for an arbitrary instantaneous change in the boundary condition. The more general case in which several wave modes exist at the same place or waves of the same mode exist in the same place propagating in opposite directions will not be considered. Problems of this kind can also be treated by a generalization of the procedures to be described; however, in most cases, they involve considerable labor.

We shall show that compression waves for both the fast and the slow modes tend to steepen to form shock waves, whereas the expansion waves for these two modes tend to spread out with time so that the gradients become less steep. The intermediate wave, on the other hand, has the rather surprising property that even for large amplitudes, it remains a linear wave. Thus, even for large amplitude, an intermediate wave of arbitrary shape will retain its shape as it propagates through the medium.

Intermediate Large Amplitude Waves

Let us imagine a semi-finite uniform plasma bounded by a piston. At time zero the piston is moved such as to produce a step function small amplitude intermediate wave. A short time later the medium will still be undisturbed ahead of the region to which the wave has propagated, i.e., for distances greater than $b_x t$ from the piston. In the region between the piston and the instantaneous location of the wave, the medium will again be uniform, but at a slightly different condition than the condition existing ahead of the wave. If at this time the piston velocity is again changed instantaneously so as to produce a second intermediate wave, we may examine the propagation speed of this second wave. In order to do this we must determine the conditions behind the first wave. Since, as we concluded in the previous section, there is no change in density, normal component of velocity, or normal component of magnetic field across an intermediate wave, the propagation speed for intermediate waves remains unchanged. Thus, the second wave will move at precisely the same speed as the first wave. We may now consider a third and fourth wave generated by the piston, and it follows from the same argument that the propagation speeds of all of these waves will be precisely equal. Since we can consider an arbitrary pulse of intermediate waves to be composed of a series of step functions, it follows that provided that the piston motion is constrained to produce only intermediate waves, the wave shape will be retained as the entire large amplitude disturbance propagates through the fluid. Thus we obtain neither steepening to form a shock wave nor spreading out as in the case of expansion fans.

The restriction on the piston motion which is required to produce a pure intermediate wave is easily seen from the condition that the change in velocity across a small amplitude intermediate wave must be perpendicular to the plane defined by the magnetic field and the wave or piston normals. Thus, the instantaneous changes in velocity or acceleration of the piston must always be perpendicular to the magnetic field at the surface of the piston.

The changes in flow properties across a large amplitude intermediate wave are obtained by summing the changes across each of the component small step function waves, which in turn are considered as differential elements. It follows immediately from Eq. (II-15) that across the large amplitude wave the changes in normal velocity, density and pressure will be zero. In evaluating the change in magnetic field we must remember that our coordinate system was chosen such that B_z was zero ahead of each small amplitude wave. Equation (II-15) therefore states that the differential change in magnetic field is in the plane of the wave front and perpendicular to the local field. Integrating a number of such changes gives the result that the magnitude of the magnetic field is unchanged across a large amplitude intermediate wave, however, the magnetic field vector can be rotated through an arbitrary large angle about an axis perpendicular to the wave front. The change in tangential velocity across the wave is from Eq. (II-15) in the direction of the change in magnetic field and is equal to $\Delta \underset{\sim}{B}/\sqrt{4\pi\rho}$. Although such a wave produces no change in the thermodynamic quantities, the normal velocity, or the magnitude of the magnetic field, it is still a large amplitude wave in the sense that the angle of rotation of the magnetic field and the change in tangential velocity can be large, i.e. of the order of radians and the propagation speed respectively.

We may anticipate that, since for small amplitude fast and slow waves the magnetic field remains in the plane defined by the wave normal and the magnetic field ahead of the wave, it will also remain in this plane for large amplitude fast and slow waves. The intermediate wave will therefore be required in flow fields in which the boundary conditions require a rotation of the plane of the magnetic field. The particular case of rotation through 180° is frequently overlooked. In this case the magnetic field appears to stay in the same plane but its tangential component changes sign. As we shall see, neither fast or slow expansion waves or shock waves can change the sign of the tangential component thus the intermediate wave will also appear in cases where such a sign change is required by the boundary conditions.